EMBEDDINGS OF PLANE CONTINUA
AND THE FIXED POINT PROPERTY

BY

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A long-outstanding problem in plane topology is the following: Do nonseparating plane continua have the fixed point property for continuous maps (Problem 107 in the Scottish Problem Book)? A number of partial solutions are known. One important result is a theorem due independently to H. Bell and K. Sieklucki which has the following consequence (pointed out by B. Brechner): A nonseparating plane continuum admitting a fixed point free map must have a Lake-of-Wada channel in its boundary in every embedding into the plane. This consequence, which seems to reveal much about the structure of any potential counter-example to the fixed point property, motivates the studies of embeddings of plane continua in this essay. The principal tool we utilize to study such embeddings is prime end theory.

Major divisions of this dissertation are devoted to

(1) Embeddings of chainable continua with (one or more) Lake-of-Wada channels, and theorems concerning the conditions under which a chainable continuum can be (re-)embedded without a Lake-of-Wada channel.

(2) An example of an atriodic nonchainable plane continuum, embedded with a Lake-of-Wada channel, of which type any minimal tree-like counter-example to the fixed point property (should one exist) is likely to be.

(3) Embeddings of triodic tree-like continua with (multiple) Lake-of-Wada channels.

(4) Inequivalent embeddings of plane continua and the prime end structures of such embeddings, including the fact that the sin 1/x continuum and the Knaster U-continuum each have uncountably many inequivalent embeddings with the same prime end structure and the same set of accessible points.

(5) The relationship between the prime end structures of a tree-like indecomposable continuum and the quotient continuum induced by shrinking each of a collection of proper subcontinua to a point.

(6) The utilization of the latter to extend theorems of W. Lewis concerning the pseudo arc, and of J. Krasinkiewicz and S. Mazurkiewicz concerning accessibility of points, subcontinua, and composants of indecomposable plane continua.

CHAPTER 1

INTRODUCTION: THE FIXED POINT PROBLEM FOR NONSEPARATING PLANE CONTINUA

1.1. Motivation

A long-standing problem in plane topology is the following: Do nonseparating plane continua have the fixed point property for continuous maps? For homeomorphisms? This question, Problem #107 in the Scottish Problem Book, credited to Sternbach, has received attention from a number of mathematicians. A survey of fixed point problems and theorems, including the above question, has been made by R.H. Bing [1969].

A number of partial solutions to the problcm are known. Chainable continua, known to be planar [Bing, 1951], have the fixed point property for continuous maps [Hamilton, 1951]. A closed disk in the plane has the fixed point property by the well-known Brouwer Fixed Point Theorem. Early known results also include that nonseparating Peano continua have the fixed point property [Borsuk, 1932], as do one dimensional nonseparating arcwise connected tree-like continua (that is trees, dendrites, and the like) [Borsuk, 1954]. More recently Hagopian showed that nonseparating arcwise connected continua [1971] and uniquely arcwise connected plane continua [1979] have the fixed point property.

Of particular interest, however, is the Cartwright Littlewood-Bell Theorem ([Cartwright and Littlewood, 1951] and [Bell, 1967]) showing that nonseparating plane continua have the fixed point property for extendable homemorphisms. It has been pointed out by B. Brechner in conversation that this result extends to essentially extendable homeomorphisms: those homeomorphisms of a plane continuum which can be extended to the plane under some reembedding of the continuum, even if not extendable in the given embedding. (See Brechner [1981].) This latter result suggests that questions of embeddings of continua into the plane may be of some significance in making further progress on the problem.

In the more general case of continuous maps, an important result is a consequence of a theorem due independently to H. Bell [1978] and K. Sieklucki [1968] : a nonseparating plane continuum admitting a fixed point free map must have a Lake-of-Wada channel in its boundary for every embedding into the plane (Embedding corollary 2.5 [Brechner and Mayer, 1981].) (For the original Three-Lakes of-Wada construction see Hocking and Young [1961].) The choice of topics in this essay is motivated by the Bell-Sieklucki Theorem and the Embedding Corollary, which seem to reveal much about the structure of any nonseparating plane continuum that could admit a fixed point free map.

For instance, if it were the case that every nonseparating nondegenerate plane continuum with a Lake-of-Wada channel in its boundary could be reembedded in the plane so that the channel disappeared, then the fixed point property would follow. In [1980a] B. Brechner and J. Mayer define the concept of a principal continuum: a nonseparating continuum that has a Lake-of-Wada channel in its boundary in every embedding into the plane. Thus one question of significance to the fixed point problem (asked in [1980a]) is: Do there exist principal continua?

1.2 Embeddings and Prime End Structure.

Our primary interest is in embedding problems, and our primary tool in investigating and comparing embeddings will be their prime end structures. Prime ends, a concept whose original development is due to Caratheodory [1913] , are a means of studying and organizing the approaches to the boundary of a simply connected plane domain. While the complement of a nonseparating continuum in the plane E^2 is not simply connected, the complement of such a continuum in the sphere S^2 is simply connected. As we can always insure that the embedding of a continuum into S^2 misses the point at infinity, there is no practical difference for our purposes in whether embeddings are in E^2 or S^2. Ursell and Young [1951] have pointed out that prime end theory can be developed just as well in the latter context.

Through prime end theory we can identify the set of prime ends of a nonseparating continuum with the set of points on the boundary of the unit circle B in E^2 (or S^2). For basic concepts and references for prime end theory, we refer the reader to Brechner [1978]. A sketch of essential concepts is included herein where needed in the various chapters. (See 2.2.8 and 4.2.1.)

A continuum with a Lake-of-Wada channel in its boundary must have a particular kind of prime end associated with it (Theorem 2.9 in Brechner and Mayer [1980a].) The arrangement of points on the boundary of the unit circle and their correspondence to kinds of prime ends determine the prime end structure of a continuum in S^2. Thus we can detect Lake-of-Wada channels through prime and structure.

Furthermore, it is shown in Brechner and Mayer [1980a] that if two embeddings of a continuum into S^2 are equivalent (in the sense that there is some homeomorphism of S^2 onto itself carrying one embedding onto the other) then the two embeddings have the same prime end structure (Theorem 2.11 of [1980a].) However, in passing from the embedding itself to the prime end structure, some information is lost. The converse of the above theorem fails, as we show in Chapter 4.

1.3 Lake-of-Wada Channels.

In succeeding chapters, a Lake-of-Wada channel running throughout the entire boundary of a continuum is called a simple dense canal, a concept due to Sieklucki [1968] , and reiterated in Brechner and Mayer [1980a], Definition 2.1. The precise definition is given herein in Section 2.2.9. Roughly, a simple dense canal (s-d-c) is a ray in the complement that converges to the entire boundary of a nonseparating continuum in S^2 in such a way that the ray goes down an ever-narrowing Lake-of-Wada channel in the boundary. We call an embedding of a continuum with a s-d-c in the complement a principal embedding.

In Chapter 2 we show that chainable continua, if indecomposable, can always be embedded in S^2 with a s-d-c. (No continuum with decomposable boundary can be embedded with a s-d-c in the boundary.) Also, if an indecomposable chainable continuum has at least one endpoint, it can be embedded without a s-d-c, that is, nonprincipally embedded. Chapter 2 also includes a wealth of examples of chainable continua embedded with one or more s-d-c's, including countably many and uncountably many. Therein and throughout liberal use is made of figures and diagrams to graphically represent the continua constructed. For us, the study of embeddings of plane continua has a significant visual component, at least on the intuitive level.

In Chapter 3 we present an example of an atriodic nonchainable continuum in S^2 embedded with a s-d-c. The example presented is of the type of tree-like plane continuum most likely to admit a fixed point free map, if any do. The likelihood is suggested by the properties the example shares with the tree-like continua admitting fixed point free maps recently constructed by D. Bellamy [1980]. Bellamy's example is trodic and nonplanar. The Fugate-Mohler modification of his example is atriodic, but it is not known if it is planar or if it is nonplanar.

In Chapter 4 we show that both the sin 1/x continuum and the Knaster U-continuum (or bucket-handle) have uncountably many inequivalent embeddings into the plane with the same prime end structure. This shows that an equivalence of prime end structures, even between homeomorphic continua, cannot be "lifted" to an equivalence of the embeddings.

In Chapter 5 we investigate the relationship between the prime end structure of a nonseparating continuum in S^2 and the prime end structure of certain quotient continua in S^2 formed by shrinking subcontinua of the given continuum to points. This enables us to extend certain theorems in Chapter 2; in particular, we show that no chainable continuum with an end subcontinuum (not equivalent to endpoint) is principal. We are also able to extend theorems of Mazurkiewicz and Krasinkiewicz regarding accessibility of composants of indecomposable plane continua.

In Chapter 6 we give several examples of triodic continua embedded in S^2 with s-d-c's. Such continua as we construct exhibit periodic homemorphisms interchanging the channels. By extending examples of W. Lewis [1981] we show that certain chainable continua also have this property.

Of the articles referenced above, Brechner [1978] and Brechner and Mayer [1980a] are essential background to the topics and treatment of this essay. With that proviso, the following chapters are relatively independent, though all interconnected by the common themes of the fixed point problem and of embeddings of plane continua. Essential definitions, theorems, and concepts are reviewed in each chapter to the extent that they will be utilized in the development of that chapter. This results in some replication of key definitions and important theorems.

CHAPTER 2
EMBEDDINGS AND PRIME END STRUCTURE OF CHAINABLE CONTINUA

2.1 Introduction.

In this chapter we apply several concepts arising out of a promising approach to the fixed point problem for nonseparating plane continua to chainable continua and their embeddings into the plane. The concepts of principal embedding and principal continuum were introduced by Brechner and Mayer [1980a, 1980b]. In the present chapter they are applied to show two main results:

2.1.1 Theorem.

Every indecomposable chainable continuum can be principally embedded into E^2.

2.1.2 Theorem.

No chainable continuum with at least one endpoint is principal.

It follows from a theorem proved independently by Bell [1978] and Sieklucki [1968] that any nonseparating continuum X which admits a fixed point free map contains a minimally invariant indecomposable subcontinuum X_0 in its boundary, and furthermore, the continuum X_0 formed by the union of X_0 and its bounded complementary domains, if any, must have the property that every embedding of it into the plane has a Lake-of-Wada channel. We call a Lake-of-Wada channel a simple dense canal (s-d-c). (See 2.2.9 for precise definition.) Such an embedding is principal. (See Embedding Corollary 2.5 of Brechner and Mayer [1980a].) The concept of a simple dense canal is due to Sieklucki [1968].

In Section 2, embeddings, both principal and nonprincipal, of the Knaster U-continuum are discussed. Lemmas 2.2.5 and 2.2.6 are instrumental in the proofs of Theorems 2.1.1 and 2.1.2.

In Section 2.3, Theorem 2.1.1 is proved. The principal embedding of the U-continuum discussed in 2.2.11 is the model for the principal embedding of any indecomposable chainable continuum.

In Section 2.4, Theorem 2.1.2 is proved. Techniques similar to those of Section 2.3 are used, but with the nonprincipal embedding of the U-continuum discussed in 2.2.7 and 2.2.8 as the model. We also raise several questions regarding strenghthenings of Theorems 2.1.1 and 2.1.2.

In Section 2.5, examples are given for each n, $1 \leq n$ and $n = 2^{\omega_0}$ of a chainable continuum M_n embedded with exactly n simple dense canals. These continua have exactly one endpoint and can be reembedded without s-d-c's in accordance with Theorem 2.1.2. Similar examples of chainable continua with multiple endpoints and with no endpoints are given. We make extensive use of "inverse-limit-with-embedding" diagrams, introduced in Brechner and Mayer [1980a] and here discussed in 2.2.7, in presenting these examples.

In what follows, all spaces are metric and all continua are compact connected spaces, and for convenience, nondegenerate.

2.2 Embeddings Of The Knaster U-Continuum.

Our technique in proving Theorems 2.1.1 and 2.1.2 will be to show that any indecomposable chainable continuum can be embedded in the plane as the intersection of a defining sequence $\{D_i\}_{i=1}^{\infty}$ of chains of open disks such that the pattern that D_i follows in D_{i-1} can be consolidated to the pattern that one chain of a defining sequence for the Knaster U-continuum follows in its predecessor. (Since we shall often refer to sequences indexed by the positive integers, we shall write them as $\{D_i\}_i$, omitting the limits on the index.) The consolidation can be done in such a way that the diameter of the links of the consolidations goes to zero. That some such consolidations of defining sequences for any indecomposable continuum exist follows from Bellamy [1981] and Rogers [1970], but for the diameter of the links to go to zero requires a stronger condition than indecomposability; chainability suffices. The definitions of chains, links, and consolidations of chains are standard. (See Bing, [1948].)

2.2.1 Definition.

A defining sequence $\{G_i\}_i$ for a continuum X is a sequence of open covers of X such that for each i, (1) G_i is coherent, (2) G_{i+1} closure refines G_i, (3) the mesh of G_i is less than $1/2^i$, (4) G_i minimally covers X, and (5) $X = \bigcap_{i=1}^{\infty} G_i^*$. ($G_i^*$ denotes the union of the links of G_i.) This definition is due to Ingram and Cook [1967], though we have strengthened condition (3).

2.2.2 Theorem.

A continuum X is indecomposable iff there is a defining sequence $\{G_i\}_i$ for X such that for each i there is a $j>i$ such that if $G_j = L_1 \cup L_2, L_1$ and L_2 being coherent subcollections of G_j, then either L_1^* or L_2^* meets every set in G_i [Ingram and Cook, 1967].

2.2.3 Definition.

In order to describe an embedding of a chainable continuum as the intersection of a defining sequence of chains of open disks in the plane, we make use of the notion of the pattern that one chain follows in another which it refines due to Bing [1948] . Roughly, a pattern is a map which tells us in which links of the containing chain the links of the refining chain sit. Thus for chains $C = \{C(1), C(2), \ldots, C(m)\}$ and $D = \{D(1), D(2), \ldots, D(n)\}$, with D refining C, the pattern D follows in C is a set of ordered pairs $\{(x_1,y_1), \ldots, (x_n,y_n)\}$ which indicates that the $x_i{}^{\text{th}}$ link of D is a subset of the $y_i{}^{\text{th}}$ link of C. A pattern is not necessarily unique, since a link of D may lie in the intersection of two links of C. Consecutive y_i's may not differ by more than one. We say a pattern reverses when for $i \geq 2$, from x_i to x_{i+1}, both $y_{i+1} < y_i$ and $y_i \geq y_{i-1}$ or both $y_{i+1} > y_i$ and y_i and $y_i \leq y_{i-1}$. A reversal in a previously constant or increasing (decreasing) immediately preceding part of the pattern represents a bend in the chain. Because patterns are not unique, the same chaining may correspond to both a pattern with and a pattern without a reversal. Such "pseudo-bends" occur in the intersection of two links of the containing chain.

2.2.4 Definitions.

A pattern alone does not determine an embedding. In order to distinguish various embeddings of the same pattern, we define a pair of notions due to Brechner [1978] and expand some upon them. Let $C = \{C(1), \ldots, C(m)\}$ and $D = \{D(1), \ldots, D(n)\}$ be chains of open disks in the plane with D refining C. By open disk we mean a convex open set with simple closed curve boundary. We assume chain C has been "straightened out" by some orientation-preserving homeomorphism of the plane so that C(1) is on the left and C(m) is on the right with the nerve of C parallel to the horizontal axis. Thus the terms "above" and "below" used below are unambiguous. Suppose D(1) is a subset of C(k). Then D is descending in C provided that from C(k), D follows a pattern in C in such a way that each time the pattern reverses (a bend), the subsequent subchain of D is below the preceding subchain (the bend is downward); D is ascending in C provided that from C(k), D follows a pattern in C in such a way that each time the pattern reverses, the subsequent subchain of D is above the preceding subchain (the bend is upward).

If D follows in C a pattern which contains no reversals, then we say D is straight through C. If D consists of two straight through subchains, one from (and contained in) the first (last) link of C to (and contained in) the last (first) link of C, and the other reversing in the last (first) link of C and returning to the first (last) link of C, then we say D is two-to-one in C. If D can be consolidated to a chain following a two-to-one pattern in C, then we say D is at least two-to-one in C. By extension of the above, if D consists of n straight through subchains running from endlink to endlink of C, then we say D is n-to-one in C. A chain D following a pattern in C which can be consolidated to an n-to-one pattern is said to be at least n-to-one in C.

By the term U-like we denote any at least 2^n-to-one pattern or any consolidation of such a pattern to a 2^n-to-one pattern. We also call a chainable continuum U-like if it has a defining sequence of at least 2^n-to-one chain covers. The thrust of Lemmas 2.2.5 and 2.2.6 is that all indecomposable chainable continua are U-like.

In what follows we uniformly use "n" subscripted by the index of the chain cover to denote the last link of the chain. Thus $C_i(n_i)$ denotes the last link of chain C_i.

That open disk chains can be used to produce an embedding of chainable continuum X into the plane, that is, guarantee the existence of a homeomorphism of X into the plane, is a consequence of Theorem 4 of Bing [1951] .

2.2.5 Lemma.

Let X be an indecomposable chainable continuum. Then there is a defining sequence $\{D_i\}_i$ of chain covers for X such that for each i>1, there exist positive integers $k_1^i<k_2^i<k_3^i$ such that (1) $D_i(k_1^i) \cup D_i(k_3^i) \subset D_{i-1}(1)$,

(2) $D_i(k_2^i) \subset D_{i-1}(n_{i-1})$, and (3) subchain $D_i(k_1^i,k_3^i)$ is minimal with respect to properties (1) and (2). Hence, D_i is at least two-to-one in D_{i-1}.

Proof. Let $\{C_j\}_j$ be a defining sequence of chain covers for X. Our proof proceeds by progressively selecting subsequences of the C_j's until we have a subsequence which almost meets conditions (1) and (2) above. A certain sequence of consolidations of the members of this final subsequence is then shown to meet conditions (1) through (3).

Select from $\{C_j\}_j$ a subsequence $\{C_a\}_a$ such that for all a, the mesh of C_a is less than $1/2^{a+1}$ and the closure of any two consecutive links of C_{a+1} is contained in one link of C_a. It then follows from Theorem 2.2.2 that for each a, there is a b>a such that C_b consists of two subchains $C_b(1,y_b)$ and $C_b(y_b,n_b)$ such that both subchains meet every link of C_a. Furthermore, as the C_a's are minimal covers, we may assume that $C_b(y_b)$ is a subset of an endlink of C_a, and that at least one of the subchains contains a link which is a subset of the other endlink of C_a.

If $C_b(y_b) \subset C_a(n_a)$, then let c = b. Otherwise, apply Theorem 2.2.2 again to find a c>b such that C_c consists of two subchains $c_c(1,y_c)$ and $C_c(y_c,n_c)$ such that both subchains meet every link of C_b (and so every link of C_a) and $C_c(y_c)$ is a subset of an endlink of C_b. In either case, we may now find positive integers $r_c<s_c<t_c$ and a subchain $C_c(r_c,s_c,t_c)$ such that both $C_c(r_c)$ and $C_c(t_c)$ meet $C_a(1)$ and at least one of them is a subset of $C_c(1)$, and $C_c(s_c)$ is a subset of $C_a(n_a)$.

By repeatedly following the above procedure, we may select a subsequence $\{C_i\}_i$ of $\{C_a\}_a$ such that for all $i>1$, there are positive integers $u_i<v_i<w_i$ and a subchain $C_i(u_i,v_i,w_i)$ such that both $C_i(u_i)$ and $C_i(w_i)$ meet $C_{i-1}(1)$, at least one of them being a subset of $C_{i-1}(1)$, and $C_i(v_i)$ is a subset of $C_{i-1}(n_{i-1})$.

We will produce the sequence $\{D_i\}_i$ by consolidating each C_i to a chain D_i while preserving closure refinement. Let $D_1 = C_1$. Assume $\{D_i\}_{i=1}^{j-1}$ have been defined. It follows from our selection of the mesh of each C_a, that $C_j(1) \cup C_j(2)$ lies in a single link of C_{j-1}, and so in a single link of the consolidation of C_{j-1}, namely D_{j-1}. Let $D_j(1) = C_j(1) \cup C_j(2)$, and $D_j(m) = C_{j(m+1)}$, for $2 \leq m \leq n_j-1$. The mesh of C_j guarantees that the mesh of D_j is less than $1/2^j$. Closure refinement is preserved, since $D_j(1)$ is a subset of a single link of D_{j-1}. Thus $\{D_i\}_i$ is a defining sequence for X.

For each $i > 1$, the triple u_i, v_i, w_i corresponds to a triple $r_i<s_i<t_i$ with the same properties with respect to D_{i-1} that the original triple had to C_{i-1} except that $D_i(r_i)$ and $D_i(t_i)$ are now both subsets of $D_{i-1}(1)$. Let k_1^i be the last link of subchain $D_i(r_i,s_i)$ in $D_{i-1}(1)$. Let $k_2^i = s_i$. Let k_3^i be the first link of $D_i(s_i,t_i)$ in $D_{i-1}(1)$. Then $\{D_i\}_i$ i satisfies conditions (1) through (3).

It can then be shown that for each i, D_{i+1} can be consolidated to a two-to-one chain in D_i. Hence, D_{i+1} is at least two-to-one in D_i. QED

Note that the selection of the k_j^i's in Lemma 2.5 (and in Lemma 2.2.6 to follow) is not unique.

2.2.6 Lemma.

Let X be an indecomposable chainable continuum. Then there is a defining sequence $\{D_i\}_i$ of chain covers for X such that for each i>1, and for each positive even integer m, there exist positive integers $k_1^i<k_2^i< \ldots <k_{m+1}^i$ such that (1) $D_i(k_1^i) \cup D_i(k_3^i) \cup \ldots \cup D_i(k_{m+1}^i) \subset D_{i-1}(1)$, (2) $D_i(k_2^i) \cup D_i(k_4^i) \cup \ldots \cup D_i(k_m^i) \subset D_{i-1}(n_{i-1})$, and (3) subchain $D_i(k_1^i, k_{m+1}^i)$ is minimal with respect to properties (1) and (2). Hence, D_i is at least m-to-one in D_{i-1}, and furthermore, X is U-like.

Proof. Let $\{D_j\}_j$ be a defining sequence of chain covers for X satisfying the conditions of Lemma 2.2.5. Then for all j > 1, D_j is at least two-to-one in D_{j-1}. Let p be chosen so that $2^{p-1} < m \leq 2^p$. Let $\{D_i\}_i$ be the subsequence of $\{D_j\}_j$ chosen by selecting the first, the p+1th, and at the nth stage, the (n-1)p+1th member of $\{D_j\}_j$. Then for each i > 1, D_i will be at least 2^p-to-one in D_{i-1}. But as 2^p is no less than m, each D_i is at least m-to-one in D_{i-1}. That $\{D_i\}_i$ satisfies conditions (1) through (3) can then be shown. QED

A version of Lemma 2.2.6 can be proved for odd m, though with k_j^i for odd j in $D_{i-1}(1)$ and k_j^i for even j in $D_j(n_{i-1})$, $1 \leq j \leq m+1$. That is, $D_i(k_{m+1}^i)$ lies in the last link of D_{i-1}, not the first.

2.2.7 Standard Embedding of the Knaster U-continuum.

We can describe what we will call the standard embedding of the Knaster U-continuum by a defining sequence $\{C_i\}_i$ of open disks in the plane such that C_{i+1} follows in C_i a descending two-to-one pattern of embedding. This two-to-one pattern and descending embedding is schematically represented by the "inverse-limit-with-embedding" diagram in Figure 2.1. The infinite repetition of the indicated pattern-with-embedding produces the standard embedding of the U-continuum. We suppose C_i to be straightened out to guide us in embedding C_{i+1} in C_i. A point p of a chainable continuum X is an endpoint if, and only if, for some defining sequence for X, p is the intersection of first links. Note that $p = \bigcap_{i=1}^{\infty} C_i(1)$ is the only endpoint of the U-continuum.

2.2.8 Prime End Structure Of The Standard Embedding Of The Knaster U-Continuum.

Basic definitions concerning prime ends and further references may be found in Brechner [1978] . Figure 2.2. represents the only accessible composant of the standard embedding of the U-continuum. A composant is accessible if it contains at least one accessible point. In this case the composant is accessible at every point, by an argument similar to the proof of Lemma 2.4.1.1. Note $\{Q_i\}_i$ is a chain of crosscuts of E^2-U such that $Q_i \rightarrow p$. This chain of crosscuts defines a prime end E. By I(E) we denote the impression of E, the intersection of the closures of the bounded domains of $Q_i \cup U$, over all i. (These domains are towered.) It can be shown that $I(E)=U$, and for any prime end F distinct from E, such as that defined by $\{R_i\}_i$ in Figure 2.2, $I(F)$ is a single point. The existence of at least one prime end E with $I(E)=U$ is necessitated by U's indecomposability [Brechner, 1978] .

By $P(E)$ we denote the set of principal points of prime end E, the set of point to each of which some chain of crosscuts defining E converges. It can be shown that $P(E)=\{p\}$, the endpoint of U, for E defined by $\{Q_i\}_i$ above. Such a prime end is of the second kind: $I(E)$ nondegenerate and $P(E)$ degenerate. Any other prime and F of U is of the first kind: $I(F)=P(F)$, both degenerate. For any prime end E, each of $I(E)$ and $P(E)$ is a continuum (or degenerate) with $P(E)$ a subcontinuum of $I(E)$.

2.2.9 Definition

A simple dense canal (s-d-c) of the continuum X is a ray $D \subset E^2-X$ (a 1-1 continuous image of $[0,\infty)$) such that (1) $\bar{D}-D=\text{Bd } X$, (2) at each point of D there is a transverse crosscut to X, and (3) diameter of such crosscuts goes to zero as we go to infinity on D. A crosscut Q is transverse to D at a point d if, and only if, $Q \cap D = \{d\}$, and for a sufficiently small open disk neighborhood U of D, each component of $U-Q$ contains exactly one component of $(U \cap D)-d$.

2.2.10 Theorem.

A nonseparating plane continuum X has a simple dense canal iff there exists a prime end E of E^2-X such that $I(E) = \text{Bd } X = P(E)$. (Theorem 2.9, [Brechner and Mayer, 1980a].)

2.2.11 Definitions.

A prime end E such that I(E) = P(E), both nondegenerate, is of the third kind. So a s-d-c corresponds to a prime end of the third kind for which I(E) = Bd X. For tree-like continua, the necessary and sufficient condition reduces to I(E) = X = P(E). Such embeddings of nonseparating continua are termed principal embeddings. If all the embeddings of X into the plane are principal, then X is called a principal continuum ([Brechner and Mayer, 1980a], Definition 2.6). In virtue of the standard embedding, the Knaster U-continuum is not a principal continuum.

2.2.12 Principal Embeddings of the U-Continuum.

In Remark 3.5 of Brechner and Mayer [1980a], we indicated that the U-continuum has a principal embedding. Consider the defining sequence for the U-continuum formed by taking each odd-numbered chain of the standard defining sequence. Let that be $\{C_i\}$. Note that the pattern one odd-numbered chain of the standard sequence follows in another is four-to-one. Embed the U-continuum according to the inverse-limit-with-embedding diagram of Figure 2.3(a) repeated infinitely. The hatched line in Figure 2.3(a) represents an initial arc of a ray D extendable in such a way as to be dense in U. Transverse crosscuts to U exist at each point of D, and as they can be confined to a single link, their diameter goes to zero. Hence D is a simple dense canal in this embedding of U. Note also that any prime end not corresponding to D is such that its impression is degenerate. Furthermore, in this embedding endpoint p and its composant are inaccessible. Figure 2.3(b) illustrates another principal embedding of the U-continuum, but with the endpoint accessible. This embedding has two prime ends whose impression is all of U, but only one of them corresponds to a s-d-c. An endcut to p corresponds to the other prime end.

It should be noted that the embeddings represented by Figures 2.1, 2.3(a), and 2.3(b) of the U-continuum are inequivalent. This is evident in comparing the differing accessibility of endpoint p in Figures 2.1 and 2.3(a). That 2.3(b) is also not equivalently embedded with respect to either of the other is a consequence of the differing prime end structures. (See Theorem 2.11 of Brechner and Mayer [1980a].)

2.3 Principal Embeddings Of Chainable Continua.

We now have sufficient tools to show that any indecomposable chainable continuum can be principally embedded. We will use the pattern to Figure 2.3(a) to produce an infolded U-like embedding of any indecomposable chainable continuum.

2.3.1 Proof Of Theorem 2.1.1.

Let X be an indecomposable chainable continuum. By Theorem 4 of Bing [1951], X can be realized as the intersection of a defining sequence of chains of open disks in the plane. However, we want a particular kind of embedding, one with a s-d-c in the complement.

There is a defining sequence $\{C_i\}_i$ for X satisfying the following conditions:

(1) C_i is taut: that is, any pair of nonadjacent links of C_i is a positive distance apart.

(2) If a link of C_i (other than an endlink) lies entirely in $C_{i-1}(k)$, for any k, $1 \leq k \leq n_{i-1}$, then that link is contained in a subchain of C_i of nine links lying entirely in $C_{i-1}(k)$.

(3) C_i is at least four-to-one in C_{i-1}; that is, for each $i > 1$, there exist positive integers $r_i < s_i < t_i < u_i < v_i$ such that:

(a) $C_i(r_i) \cup C_i(t_i) \cup C_i(v_i) \subset C_{i-1}(1)$,

(b) $C_i(s_i) \cup C_i(u_i) \subset C_{i-1}(n_{i-1})$,

(c) subchain $C_i(r_i, v_i)$ is minimal.

Conditions (1) and (2) follow from the fact that X is compact. Condition (3) follows from Lemma 2.2.6.

We may think of the C_i-covers abstractly. Then X can be embedded in the plane by specifying a particular defining sequence $\{D_i\}_i$ of chains of open disks in the plane whose intersection can be shown to be homeomorphic to X. Such a homeomorphism is defined in Theorem 11 of Bing [1948], which Bing uses to prove Theorem 4 of [1951]. What is required to apply Theorem 11 is that D_i follow in D_{i-1} the pattern that C_i follows in C_{i-1}.

In producing the embedding below, we think of the D_i-chains semi-abstractly as chains of topological open disks with as yet no particular embedding in the plane. We then specify how these open disks are to be situated in the plane to produce the desired embedding of X. Figure 2.4(a) is an example of the pattern that some D_i might follow in D_{i-1}. For convenience, we have written the pattern descending, but at the moment think of it abstractly, with no particular embedding.

In carrying out our embedding, we make use of an auxiliary sequence of open disk chains $\{U_i\}_i$, such that U_i refines D_{i-1} exactly four-to-one and contains D_i in a particular way. Chain $U_i(1,m_i)$ consists of four straight through subchains $U_i(1,h_i)$, $U_i(h_i,j_i)$, $U_i(j_i,k_i)$, and $U_i(k_i,m_i)$ with $1 < h_i < j_i < k_i < m_i$, and links $U_i(1)$, $U_i(j_i)$, and $U_i(m_i)$ lie in $D_{i-1}(1)$, while links $U_i(h_i)$ and $U_i(k_i)$ lie in $D_{i-1}(n_{i-1})$. Furthermore, $U_i(1,h_i)$ contains $D_i(1,r_i,s_i)$, $U_i(h_i,j_i)$ contains $D_i(s_i,t_i)$, $U_i(j_i,k_i)$ contains $D_i(t_i,u_i)$, and $U_i(k_i,m_i)$ contains $D_i(u_i,v_i,n_i)$. The links of D_i numbered r_i, s_i, t_i, u_i, v_i lie in the links of U_i numbered 1, h_i, j_i, k_i, m_i, respectively. In placing a link of D_i in a link of U_i, we preserve the pattern that D_i follows in D_{i-1}. That is, if $D_i(a) \subset U_i(b)$ and $U_i(b) \subset D_{i-1}(c)$, then $D_i(a) \subset D_{i-1}(c)$ in the original pattern. Note that $D_i(1)$ may be in any link of $U_i(1,h_i)$, and $D_i(n_i)$ may be in any link of $U_i(k_i,m_i)$, depending upon what the original pattern was.

Suppose that D_{i-1} has been realized as a chain of open disks in the plane. Then U_i is embedded in D_{i-1} with the infolded pattern-with-embedding of Figure 2.4(b). We have placed no restriction on the mesh of U_i other than that U_i refine D_{i-1}. There could be as few as four links of U_i (one from each subchain) in an interior link of D_{i-1}, essentially subdividing that interior link into four links, and as few as two links of U_i in $D_{i-1}(n_{i-1})$, just $U_i(h_i)$ and $U_i(k_i)$. However, in order to keep the links of U_i open disks, we need at least five links in $D_{i-1}(1)$: $U_i(1)$, $U_i(m_i)$, and a three-link subchain with $U_i(j_i)$ in the middle. Such a U_i-chain meeting these minimal requirements is illustrated in Figure 2.5.

We now embed subchains $D_i(r_i,s_i)$, $D_i(s_i,t_i)$, $D_i(t_iu_i)$, and $D_i(u_i,v_i)$ descending in $U_i(1,h_i)$, $U_i(h_i,j_i)$, $U_i(j_i,k_i)$, and $U_i(k_i,m_i)$, respectively. This pattern-with embedding is shown for our example in Figure 2.4(c). Compare also Figure 2.5. Note that links of D_i numbered r_i, s_i, t_i, u_i, v_i lie in links of U_i numbered 1, h_i, j_i, k_i, m_i, respectively, and that the pattern D_i follows in D_{i-1} is preserved, as can be seen by comparing Figures 2.4(a) and 2.4(d). Since there are at least nine links of D_i in the subchain lying in $D_{i-1}(1)$ that contains $D_i(t_i)$, we can require that $D_i(t_i)$ have four links of D_i both preceding and following it lying in $D_{i-1}(1)$. Hence there are enough links for D_i to refine $U_i(j_i-1,\ j_i+1) \subset D_{i-1}(1)$, and still have the diameter of links of D_i less than half the diameter of links of D_{i-1}.

The initial and final subchains of D_i are embedded with $D_i(1,r_i)$ descending $U_i(1,h_i)$ and above $D_i(r_i,s_i)$ in $U_i(1,h_i)$, and $D_i(v_i,n_i)$ ascending in $U_i(k_i,m_i)$ and above $D_i(u_i,v_i)$ in $U_i(k_i,m_i)$. The resulting pattern-with-embedding diagram for our examples, showing how D_i now sits in D_{i-1}, is illustrated in Figure 2.4(d). How the chains

D_{i-1} (straightened out), U_i, and D_i might appear in the plane is shown in Figure 2.5.

A s-d-c D may then be constructed stage-by-stage as we illustrated with the principal embedding of the U-continuum. An arc of the canal in our example is shown as the hatched line in both Figures 2.4(d) and 2.5. That $D_i(n_i)$ may not lie in $D_{i-1}(1)$, in contrast to the situation with the U-continuum, can only make this arc of the canal proportionately longer. QED

2.4 Nonprincipal Embeddings Of Chainable Continua.

For a decomposable chainable continuum, every embedding into the plane is nonprincipal, as a s-d-c in X implies Bd X is indecomposable ([Brechner and Mayer, 1980a] Theorem 2.9.) For an indecomposable chainable continuum, we can get an embedding using the two-to-one standard, rather than the infolded four-to-one, embedding pattern for the U-continuum as our model. However, while the standard embedding of the U-continuum has no s-d-c, we have as yet been able to produce such an embedding (without a s-d-c) of an indecomposable chainable continuum, in general, when the continuum has at least one endpoint. Various examples (see our 2.5.7) of chainable continua without endpoints have proved reembeddable without a s-d-c, but the general proof eludes us, though we conjecture that it can be done.

2.4.1 Proof Of Theorem 2.1.2

Let X be an indecomposable chainable continuum with endpoint p. We can find a defining sequence $\{C_i\}_i$ of chain covers of X satisfying the following conditions:

(1) p is in $C_i(1)-\overline{C_i(2)}$; hence $C_i(1) \subset C_{i-1}(1)$.

(2) C_i is taut. (As defined in 3.1(1).)

(3) If a link of C_i (other than an endlink) lies entirely in $C_{i-1}(k)$, for any k, $1 \leq k \leq n_{i-1}$, then that link is contained in a subchain of C_i of nine links lying entirely in $C_{i-1}(k)$.

(4) C_i is at least two-to-one in C_{i-1}; that is, for each $i > 1$, there exist positive integers $r_i < S_i < t_i$ such that:

(a) $C_i(r_i) \cup C_i(t_i) \subset C_{i-1}(1)$,

(b) $C_i(s_i) \subset C_{i-1}(n_{i-1})$,

(c) subchain $C_i(r_i,s_i,t_i)$ is minimal.

(5) Subchain $C_i(1,r_i)$ is minimal; that is, no link of $C_i(1,r_i)$ lies in $C_{i-1}(n_{i-1})$.

Condition (1) holds because p is an endpoint of X. See Section 5 of Bing [1951]. Conditions (2) and (3) follow from the fact that X is compact. Condition (4) is a consequence of Lemma 2.2.5. Since first links are towered, we may choose the first subchain satisfying condition (4), thus satisfying condition (5).

As in the proof of Theorem 2.2.1, our procedure for embedding X is to specify a defining sequence $\{D_i\}_i$ of chains of open disks in the plane such that D_i follows in D_{i-1} the pattern that C_i follows in C_{i-1}. We think of $\{D_i\}_i$ semi-abstractly, as chains of topological open disks with no particular embedding. Figure 2.6(a) indicates how some D_i might sit in D_{i-1}, written descending for convenience. In some cases it could be that $r_i=1$, though not in the example we illustrate in Figure 2.6(a). Also note that s_i is not uniquely determined: any of the three "loop ends" of D_i that lie in $D_{i-1}(n_{i-1})$ in Figure 2.6(a) could contain $D_i(s_i)$.

In carrying out our embedding we make use of an auxiliary sequence $\{U_i\}_i$ of chains of open disks, such that U_i refines D_{i-1} exactly two-to-one and contains D_i in a particular way. Chain $U_i(1,m_i)$ consists of two straight through subchains $U_i(1,k_i)$ and $U_i(k_i,m_i)$ with $1 < k_i < m_i$, and links $U_i(1)$ and $U_i(m_i)$ lie in $D_{i-1}(1)$, while link $U_i(k_i)$ lies in $D_{i-1}(n_{i-1})$. Furthermore, $U_i(1,k_i)$ contains $D_i(1,r_i,s_i)$ and $U_i(k_i,m_i)$ contains $D_i(s_i,t_i,n_i)$. The links of D_i numbered 1, r_i, s_i, t_i lie in the links of U_i numbered 1, 1, k_i, m_i, respectively. In placing a link of D_i in a link of U_i, we preserve the pattern that D_i follows in D_{i-1}. That is, if $D_i(a) \subset U_i(b)$ and $U_i(b) \subset D_{i-1}(c)$, then $D_i(a) \subset D_{i-1}(c)$ in the original pattern. Note that $D_i(1)$ may be in any link of $U_i(1,h_i)$, and $D_i(n_i)$ may be in any link of $U_i(k_i,m_i)$, depending upon what the original pattern was.

Suppose that D_{i-1} has been realized as a chain of open disks in the plane. We embed U_i in D_{i-1} following the descending pattern-with-embedding of Figure 2.6(b). We then embed subchains $D_i(1,r_i,s_i)$ and $D_i(s_i,t_i)$ descending in $U_i(1,k_i)$ and $U_i(k_i,m_i)$, respectively. This pattern-with-embedding is shown for our example in Figure 2.6(c). Note that links of D_i numbered 1, r_i, s_i, t_i lie in links of U_i numbered 1, 1, k_i, m_i, respectively, and that that pattern D_i follows in D_{i-1} is preserved. We embed the final subchain $D_i(t_i,n_i)$ in $U_i(k_i,m_i)$ descending and below $D_i(s_i,t_i)$ in $U_i(k_i,m_i)$. Note that D_i is descending throughout U_i. Figure 6(d) illustrates the resulting embedding of D_i in D_{i-1}. Note that $D_i(1,s_i)$ is descending in D_{i-1} and that the initial subchain of D_{i+1} will be descending in $D_i(1,s_i)$.

Figure 2.7 represents the first three stages of a typical continuum X embedded according to the above procedure. The intervening U_i chains have been omitted. The heavier line is the minimal subchain $D_3(r_3,s_3,t_3)$.

It is the embedding of X constructed above that we claim has no simple dense canal. We prove this claim by showing that only one prime end E, defined by a chain of crosscuts converging to endpoint $p = \bigcap_{i=1}^{\infty} D_i(1) = \bigcap_{i=1}^{\infty} U_i(1)$, is such that $I(E) = X$. But, as p is accessible with respect to E, $P(E) = \{p\}$, so E is not of the third kind, so does not correspond to a s-d-c. We will show that every prime end F distinct from E is such that I(F) is a proper subcontinuum of X, so even if F is of the third kind, it does not correspond to a s-d-c. (It is possible for a point, even an endpoint, to be accessible with respect to one prime end and also be a principal point of a distinct prime end, including a prime end corresponding to a s-d-c, as is illustrated in Figure 2.3(b) for a principal embedding of the U-continuum. Example 2.5.1 also has this property. Hence we cannot simply assume that a chain of crosscuts defining prime end F does not converge to p.)

Our proof is similar to those of Theorems 4.3.1 through 4.3.3 of Brechner [1978]. It will be convenient and aid comparison to divide our proof into three lemmas roughly corresponding to Brechner's theorems.

2.4.1.1 Lemma. All accessible points of X are in the composant C_p of endpoint p.

Proof. Assume D_1 is straightened out and let R and S be rays in the plane drawn respectively upward and downward from endpoint p of X so that R (respectively, S) meets Bd $D_i(1)$ in exactly one point and otherwise meets no other link of D_1. Then $R \cup S$ separates E^2 into right and left half-planes with X contained in the closure of the right half-plane, and only p in X in the closure of the left half-plane. (See Figure 2.7.)

Let q be any accessible point of X distinct from p. There are two homotopy classes of crosscuts in the right half-plane minus X from p to q: those like T_1 in Figure 2.7 which go around X from p to q, and those like T_2 in Figure 2.7 which do not go around X. Let A be a crosscut from p to q which, for some k, is homotopic with fixed endpoints in the right half-plane minus X to a crosscut which, except for short terminal segments in $D_k(1)$ and the link of D_k containing q, lies entirely above D_k in the right half-plane when D_k is straightened out. We can do this because A can be chosen so that its image under a straightening-out homeomorphism lies above some straightened out D_k chain. We may assume that for all i, A meets only those links of D_i that contain p and q.

Let K be the subcontinuum of X irreducible between p and q. Let C_i be the subchain of D_i which minimally covers K. The first links of $\{C_i\}_i$ are towered on p. Since A enters a link of D_k from above, it meets a link of $D_{k+1}(1,s_{k+1})$. But $D_{k+1}(1,s_{k+1})$ is descending in D_k and no link of D_{k+1} lies above $D_{k+1}(1,s_{k+1})$. All subsequent initial subchains of $\{D_i\}_{i=k+2}^{\infty}$ that lie in $D_{k+1}(1,s_{k+1})$ are descending in their immediately containing chain, and at each stage, no subsequent subchain lies above them. Hence, the last links of $\{C_i\}_i$ are towered on q. Though $C_i(1)$ is in every case the first link of D_i, the last link of C_i is not the last link of D_i for all $i > k$. Indeed, $\bigcap_{i=1}^{\infty} C_i$ will be at least some fixed distance d from any point of X in $D_{k+1}(n_{k+1})$. So $K = \bigcap_{i=1}^{\infty} C_i \neq X$. Hence, p and q lie in a proper subcontinuum of X, and so are in the same composant.
QED (2.4.1.1)

Endpoint p is accessible, and therefore, a prime end E corresponding to an endcut to p is defined by any chain of crosscuts $\{Q_i\}_i$ which cuts R and S in exactly one point each (or is homotopic to such a chain). Any such chain converges to p, so p is the only principal point of E. In Figure 2.7, Q_1 and Q_2 illustrate such a chain. Since E has only a single principal point, E does not correspond to a s-d-c.

2.4.1.2 Lemma. If F is a prime end of X distinct from E, then I(F) is a proper subcontinuum of X (indeed, of C_p.)

Proof. Let F be a prime end of X distinct from E and $\{P_i\}_i$ a chain of crosscuts defining F. If some subchain $\{P_j\}_j$ of $\{P_i\}_i$ is such that P_j intersects R and S exactly once each (or is homotopic with fixed endpoints in E^2-X, to such a crosscut), then $\{P_j\}_j$ defines prime end E. So there is a J, such that for all $i \geq J$, P_i is homotopic with fixed endpoints in E^2-X, to a crosscut that does not intersect $R \cup S$. We may assume that J=1, and that $\{P_i\}_i$ is a chain of crosscuts, none of which intersect $R \cup S$. In Figure 2.7, P_1 and P_2 illustrate such a chain of crosscuts.

$P_1 \cup X$ separates the right half-plane of E^2, and the bounded domain cut off by $P_1 \cup X$ contains P_i for all $i > 1$. Because of the U-like embedding, we may straighten out chains without disturbing $R \cup S$, so that P_1 may be seen to be homotopic to a crosscut which lies above U_k, and therefore above D_k, in the right half-plane, for some k. We may assume P_1 meets only those links of U_i and D_i that contain the endpoints of P_1. Then, as in the proof that X contains only one accessible composant, the continuum irreducible between the endpoints of P_1 is a subcontinuum of the accessible composant C_p of X, and so is properly contained in X. Since the domains cut off by a chain of crosscuts are towered, I(F) is properly contained in X. QED(2.4.1.2)

In Figure 2.7, P_1 and P_2 are the first two crosscuts in a chain defining some prime end F distinct from E. Note that if the bending of subsequent chains D_i, for $i > 1$, is back toward endpoint p and descending, as in D_2 and D_3, then the continuum irreducible between the endpoints of P_1 may contain endpoint p. This is the case if X is the pseudo arc [Brechner, 1978].

2.4.1.3 Lemma. There is only one prime end E of X such that I(E) = X, but X contains no s-d-c.

Proof. From Lemma 2.4.1.2 it follows that no prime end F distinct from E corresponds to a s-d-c. We have already observed that E does not correspond to a s-d-c. As X is indecomposable, some prime end must be such that its impression is X. Only E qualifies, so I(E) = X, but X contains no s-d-c. QED(2.4.1.3)

With Lemma 2.4.1.3 we conclude the proof of Theorem 2.1.2, having shown that every chainable continuum with at least one endpoint can be nonprincipally embedded in E^2. It would be welcome if the condition that the continuum have an endpoint could be eliminated.

2.4.2 Question.

Can every chainable continuum be nonprincipally embedded?

If so, then the fixed point property for chainable continua would follow as a corollary to Embedding Corollary 2.5 of Brechner and Mayer [1980a].

2.4.3 Question.

Can every indecomposable tree-like plane continuum be principally embedded?

2.4.4 Question.

Can every tree-like plane continuum be nonprincipally embedded?

If so, then all tree-like plane continua have the fixed point property. We conjecture, though we have no proof, that the atriodic nonchainable continuum in Mayer, [1980b] and in Chapter 3 is a principal continuum. If our conjecture is true, then the answer to 2.4.4 is no.

2.5 N-Principal Embeddings Of Chainable Continua.

An embedding of continuum X into the plane with exactly n s-d-c's is termed an n-principal embedding. If every embedding of X into the plane contains at least n s-d-c's, then X is an n-principal continuum ([Brechner and Mayer, 1980a], Definition 2.6.) In this section we present examples of chainable continua with n-principal embeddings.

Details of the proofs that our examples have the properties we mention are left to the reader. Figures are provided for the most part, and results stated are generally evident from the figures and the techniques used in the proofs of Theorems 2.1.1 and 2.1.2.

2.5.2 Example

A chainable continuum M_n can be constructed for each $n \geq 0$ embedded with exactly n s-d-c's. The Knaster U-continuum in an embedding in which two-to-one chains are alternately ascending and descending is M_0. For each $n > 0$, M_n is defined by a sequence of chains (of open disks in the plane), each member of which consists of $2(n+1)$ straight through subchains (though not all straight through the entire containing chain). A pair of consecutive subchains comprises a loop, which consists of a straight through chain from the first link of the containing chain to a designated link, then returning straight through to the first link. There is one "longest" loop which is straight through from end to end of the containing chain, succeeded by n "shorter" loops, of graduated "length." The embedding alternates ascending and descending versions of the above pattern. Figure 2.8(a) illustrates the inverse-limit-with-embedding diagram for M_2. Note the alternation of an ascending with a descending pattern. The infinite repetition of this pair of embedding patterns produces M_2. Note that the bends of C_3 occur in bend or end links of C_2 that lie in $C_1(1)$. It can be shown that M_2 (and M_n, for any n) has exactly one endpoint which is the limit of all bend and end links. To show that there is just one endpoint, show that every other point is an interior point of an arc; that is, is in an interior link at every stage of a defining subsequence of straight through subchains.

Figure 2.8(c) illustrates several stages in the chain construction of M_2 in the plane. There, and in Figure 2.8(a), the lines marked with differing symbols represent the two s-d-c's. Note that a segment of a s-d-c passing through links of C_2 and between upper and lower subchains of C_3 meets every link of C_1. In this fashion one can show the canal is dense in M_2. Figure 2.8(b) shows how to reembed M_2 with no s-d-c's in accordance with Theorem 2.1.2.

In the 2-principal embedding of M_2 (and the n-principal embedding of any M_n) the endpoint is the only accessible point of its composant. This is because the chains are alternately ascending and descending, and so "block" endcuts to other points of the endpoint composant. The endpoint is also a principal point of every prime end corresponding to in Figure 2.10 of a chainable continuum M_c embedded with c one of the s-d-c's. Note that an endcut to the endpoint also corresponds to a prime end whose impression is all of M_2.

2.5.2 Example.

We can modify the construction in Example 2.5.1 to produce a chainable continuum M_{ω_0} embedded with a countable infinity of s-d-c's. At each stage in the sequence of chains defining M_{ω_0} there is one more loop than in the immediately preceding stage. Figure 2.9 illustrates the inverse-limit-with-embedding diagram for the ω_0-principal embedding. Note that chains are alternately ascending and descending. In accordance with Theorem 2.1.2, M_{ω_0} can be reembedded with no s-d-c's. The required reembedding is similar to that of M_2 in Figures 2.8(b).

2.5.3 Example.

A chainable continuum with $c = 2^{\omega_0}$ s-d-c's can be constructed in a fashion similar to that of Example 2.5.2 by adding at each stage an additional loop for every loop of the preceding stage. We illustrate a different construction s-d-c's. In general, chain C_i contains $2^{i-1}+1$ loops, as illustrated. Chains are uniformly descending, but alternate starting in link 1, link n_{i+1}, link n_{i+2}, link 1, every four stages. Only three of the s-d-c's are illustrated by the lines marked with differing symbols in Figure 2.10. That the number of s-d-c's is 2^{ω_0} can be seen by noting that they can be represented as the branches of an infinite binary tree. By reembedding M_c with the longest loop on the outside at each stage, all s-d-c's are eliminated. This particular reembedding does not follow the proof of Theorem 2.1.2.

2.5.4 Example.

The special embedding P_s of the pseudo arc is produced by alternating descending with ascending crooked chains between opposite endpoints p and q of P_s. In [1978] Brechner describes this embedding and conjectures that each accessible point of P_s lies in a different composant. In [1980] Lewis shows this conjecture to be true. The endpoints of any crosscut of P_s thus lie in different composants. Hence the continuum irreducible between the endpoints is P_s. It follows that for any prime end E of P_s, $I(E)=P_s$.

2.5.4.1 Theorem. The special embedding P_s of the pseudo arc contains uncountably many s-d-c's.

Proof. Let $\{C_i\}_i$ be a defining sequence of crooked, alternately ascending and descending, open disk chain covers for the special embedding P_s of the pseudo arc. We show that a binary tree can be constructed dense in P_s. Each branch corresponds to a s-d-c. We first show that a ray can be constructed dense in P_s and forming a s-d-c, and then show that countably many branch points can be introduced to the ray, turning it into an infinite tree.

Suppose ray R enters link $C_i(a_i)$ of chain C_i. Without loss of generality we assume R enters C_i from above (viewing C_i as straightened out) and that C_{i+1} is descending in C_i. Since C_{i+1} is crooked in C_i, there is a (crooked) loop extending from $C_i(1)$ to $C_i(a_i)$ and back to $C_i(2)$, and the loop is followed by a subchain returning to $C_i(a_i)$. Extend R through links $C_i(2,a_i)$ (R is moving in reverse order) passing between the lower part of the loop and the returning subchain, and entering a link $C_{i+1}(a_{i+1})$ sitting in $C_i(2)$. This situation, with only a few of the bends of C_{i+1} shown, is diagramed in Figure 2.11(a). Now C_{i+2} is ascending in C_{i+1}. In link $C_{i+1}(a_{i+1})$ we can find a loop-end, where the loop of C_{i+2} extends from $C_{i+1}(n_{i+1})$ to $C_{i+1}(a_{i+1})$ and back to $C_{i+1}(n_{i+1}-1)$, and the loop is followed by a subchain returning to $C_{i+1}(a_{i+1})$ (all in reverse order). We can extend R through links $C_{i+1}(a_{i+1},n_{i+1}-1)$ between links of the lower part of the loop and the returning subchain, and entering a link $C_{i+2}(a_{i+2})$ that lies in $C_{i+1}(n_{i+1}-1)$. Note that R reaches within twice the mesh of C_i of p and within twice the mesh of C_{i+1} of q. Proceeding in this fashion, we can extend R alternately toward p and q, and so that the closure of R includes P_s. We have constructed R so that crosscuts of decreasing diameter exist at every point, as we go to infinity on R; hence R constitutes a s-d-c in P_s.

To see that there are uncountably many s-d-c's in P_s, we modify our construction above. Assume that above we pick the first loop we come to that fits the described conditions. (For descending chains this will be the first loop satisfying the conditions in the order on the chain; for ascending chains it will be the last loop in that order.) Note that in link $C_{i+1}(a_{i+1}-1)$ and below the loop ending in $C_{i+1}(a_{i+1})$ there is a loop, possibly consisting of our previously designated returning subchain and a subchain from $C_{i+1}(a_{i+1}-1)$ to $C_{i+1}(n_{i+1}-1)$, with still another returning subchain back to $C_{i+1}(a_{i+1}-1)$ below it. We may introduce a branch point to R in link $C_{i+1}(a_{i+1})$ and extend a branch R' from R at that point to $C_{i+1}(a_{i+1}-1)$ and then further extend R' through links $C_{i+1}(a_{i+1}-1, n_{i+1}-1)$ between the loop and the returning subchain and entering a link $C_{i+2}(b_{i+2})$ that sits in $C_{i+1}(n_{i+1}-1)$. This link of C_{i+2} will be distinct from that which R entered, and R and R' follow separate "channels" between links of C_{i+2}. We may repeat this branching maneuver in both $C_{i+2}(a_{i+2})$ and $C_{i+2}(b_{i+2})$. Figures 2.11(b) and (c) indicate one branching and the continuation of the rays in separate channels thereafter. Figure 2.11(d) illustrates an entire crooked refinement (with minimal bending). Each link (3 to n_i-1) of the containing chain is the entrance to a distinct channel, from top (the hatched lines) and bottom (where we show an example of branching). By repeatedly branching, we can extend R so as to form a binary tree dense in P_s, so P_s has uncountably many s-d-c's. QED

2.5.5 Example.

Theorem 3.1 of Brechner and Mayer [1980a] shows that the three-point continuum has embeddings both with and without s-d-c's. Continua with multiple s-d-c's can be produced with exactly two endpoints, with uncountably many endpoints (the pseudo arc), or with a number of endpoints proportional to the number of s-d-c's. We describe one series of examples from the many possible below.

The Knaster S-continuum (like the U-continuum in Example 2.5.1) can be used as a basis for constructing continua with multiple s-d-c's. Figure 2.12(a) is the standard embedding of the S-continuum. It has exactly two endpoints, whose composants are accessible at every point, and no s-d-c. Figure 2.12(b) is an embedding of the S-continuum with one s-d-c. (Two embedding patterns alternate.) We can produce a continuum Z_n embedded with exactly 2n s-d-c's and having exactly two endpoints. The embedding of the S-continuum with alternate chains ascending and descending is Z_0. For Z_n we introduce n extra loops of graduated length in both the first and last of the three straight through subchains defining Z_0 in a manner similar to that of Example 2.5.1. The endpoints of Z_n are opposite endpoints, since there is a definnig sequence for Z_n such that each chain is a chain from one endpoint to the other. For a continuum with a pair of opposite endpoints it is enough to embed it with all chains between the two endpoints descending to eliminate all s-d-c's. The proof is similar to that of Theorem 2.1.2; show that all but the prime ends corresponding to endcuts to the opposite endpoints have properly contained impressions.

To produce continua with n s-d-c's and two endpoints for odd n, we need only eliminate one of the loops from each stage of the construction of Z_k, $k = (n+1)/2$. Other variations on the S-continuum are also possible.

2.5.6 Example.

Bellamy [1980] has constructed an example of an indecomposable chainable continuum with no endpoints. Figure 2.13(a) is an example of such a continuum N discovered independently by us. Continuum N has no s-d-c in the given embedding, though it does have a simple canal, dense in a proper subcontinuum of N. Figure 2.13(c) represents several stages in the chain construction of N in the plane. The simple canal, not dense in N, is illustrated by the hatched line in both Figures 2.13(a) and (c). That N has no endpoints follows from the fact that every point of N is an interior point of an arc, that is, is in the intersection of a tower of straight through subchains, and always in an interior link. The proper subcontinuum N_u of N in which the indicated canal is dense is the intersection of the subchains $C_i(1,p_i)$ which are towered. Inspection reveals N_u to be homeomorphic to the U-continuum with $p = \bigcap_{i=1}^{\infty} C_i(p_i)$ as its endpoint. (Note that $C_i(p_i)$ is the link in which $C_{i+1}(1)$ and $C_{i+1}(p_{i+1})$ sit.) Though the canal in N is dense in N_u, it is *not* a s-d-c *of* N_u because there are not transverse crosscuts at each point *to* N_u.

One composant of N, inaccessible in the given embedding, consists of the union of N_u with a ray (a 1-1 continuous image of $[0,\infty)$) at their common endpoint p. If N_u is shrunk out of N, the quotient space is the U-continuum. The endpoint of the U-continuum corresponds to the nondegenerate element of the decomposition. (These properties of N were pointed out by Beverly Brechner in discussion.)

Figure 2.13(b) illustrates a modified construction producing continuum N' in which the hatched line represents a canal that *is* dense in N'. Note that link $C_i(p_i)$ now lies in the second straight through subchain. Note that the $C_i(1,p_i)$ subchains are not towered in the sequence defining N', and indeed, can be used to define an increasing tower of proper subcontinua whose limit is N'.

2.5.7 *Example*.

Continuum N, like the U- and S-continua, can be used as a basis for constructing continua embedded with multiple s-d-c's. For each $n \geq 0$, we can construct a continuum N_n embedded with exactly n s-d-c's and having no endpoints. Embedding N with alternate chains ascending and descending produces N_0. Figure 2.14(a) illustrates N_2. Compare Figures 2.14(a) and 2.8(a) for the similarity in the constructions of M_2 and N_2. Figure 2.14(b) shows how N_2 may be reembedded without a s-d-c. An endcut to $p = \bigcap_{i=1}^{\infty} C_i(p_i)$ corresponds to the only prime end whose impression is N_2. Any of the continua N_n can be similarly reembedded so as to have no s-d-c.

As is the case with $N = N_0$, each continuum N_n contains a composant which is the union of a U-continuum subcontinuum N_u and a ray at their common endpoint p. If N_u is shrunk out of N_n, the quotient space is continuum M_n of Example 2.5.1. Note that in the nonprincipal embedding of N_2 given by Figure 2.14(b), the composant containing N_u is accessible, but N_u itself is accessible only at its endpoint p. In the 2-principal embedding of N_2 given by Figure

2.14(a), the composant containing N_u is again accessible, and now N_u is accessible, though not at every point (N_u is indecomposable), and in particular, not at point p.

In the manner of Examples 2.5.2 and 2.5.3 we can also construct chainable continua N_{ω_0} and N_c with ω_0 and c s-d-c's, respectively.

Any of our examples can be constructed with crooked rather than straight through chains, thus producing embeddings of the pseudo arc with n s-d-c's, for each $n \geq 0$.

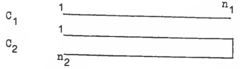


Figure 2.1

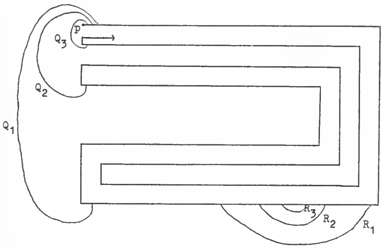
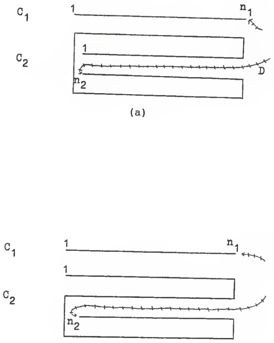


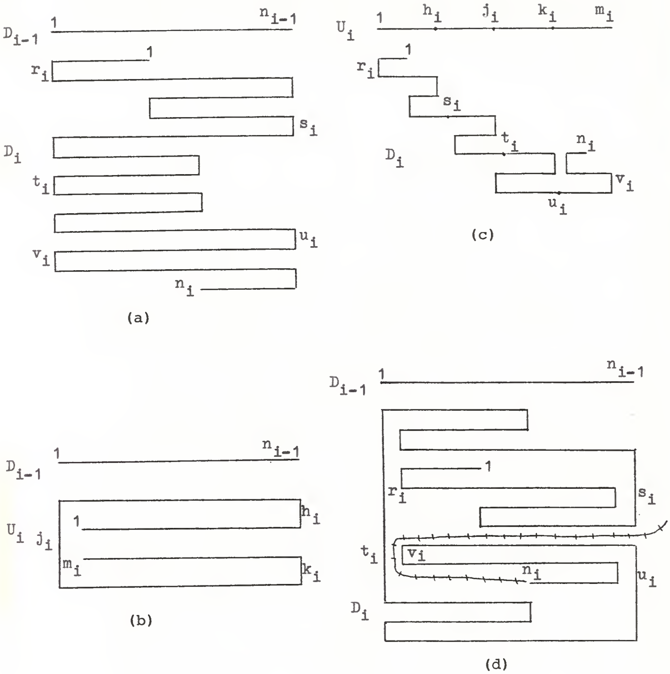
Figure 2.2



(a)

(b)

Figure 2.3



(a)

(b)

(c)

(d)

Figure 2.4

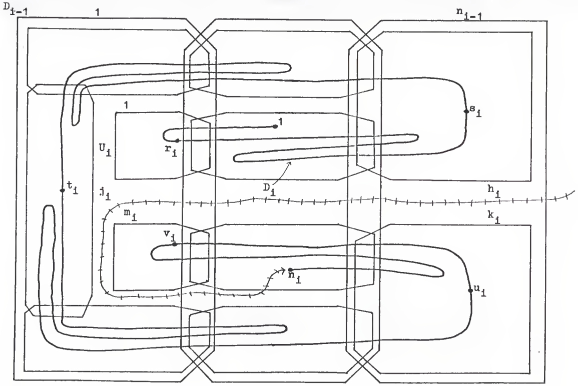


Figure 2.5

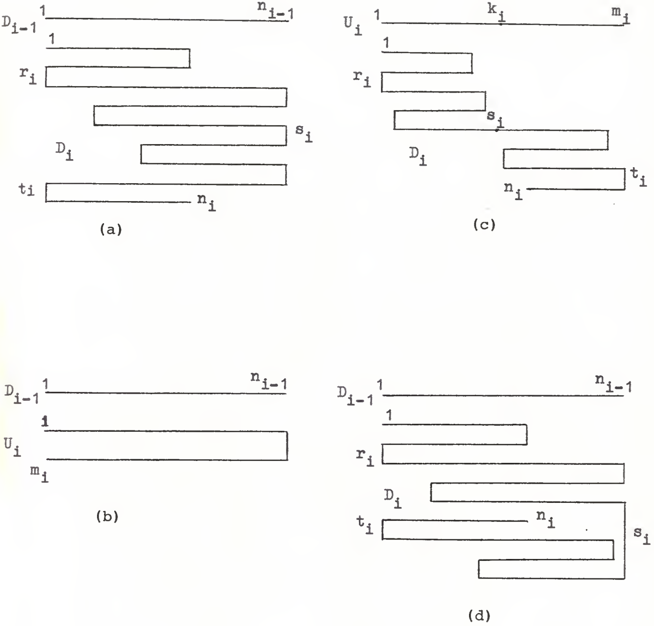

D_{i-1}
1
n_{i-1}
1
r_i
s_i
D_i
t_i
n_i
(a)
U_i
1
k_i
m_i
1
r_i
s_i
D_i
t_i
n_i
(c)
D_{i-1}
1
n_{i-1}
1
U_i
m_i
(b)
D_{i-1}
1
n_{i-1}
1
r_i
D_i
t_i
n_i
s_i
(d)

Figure 2.6

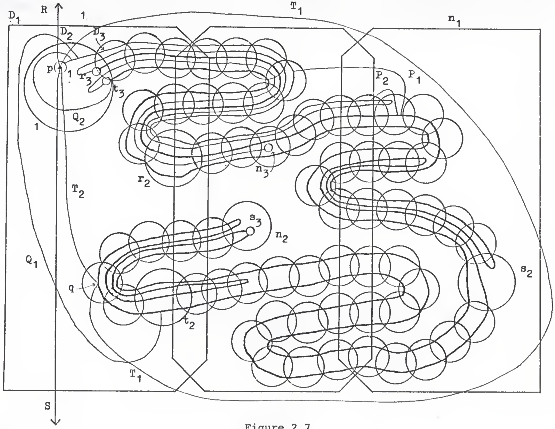
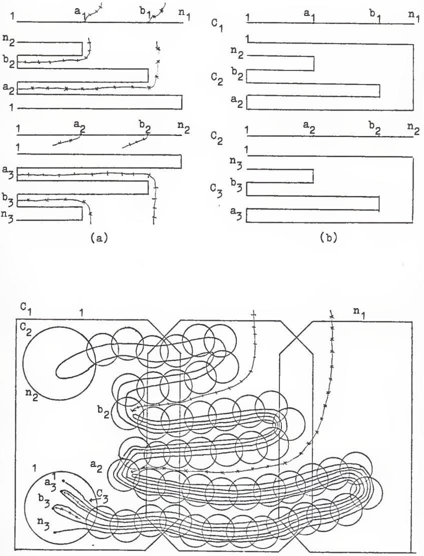


Figure 2.7



(a) (b)

(c)

Figure 2.8

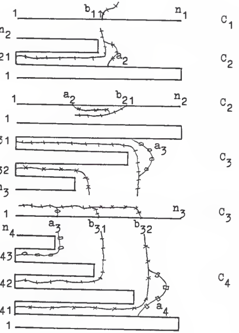
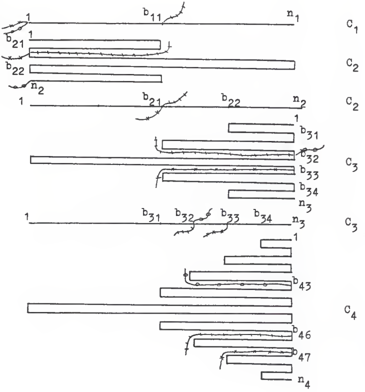

1
b_{11}
n_1
C_1
n_2
b_{21}
a_2
C_2
1
1
a_2
b_{21}
n_2
C_2
1
b_{31}
a_3
b_{32}
C_3
n_3
1
n_3
C_3
n_4
a_3
b_{31}
b_{32}
b_{43}
b_{42}
C_4
b_{41}
a_4
1

Figure 2.9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b11
n1
C1
b21
1
C2
b22
n2
1
b21
b22
n2
C2
1
b31
b32
C3
b33
b34
n3
1
b31
b32
b33
b34
n3
C3
1
b43
C4
b46
b47
n4

Figure 2.10

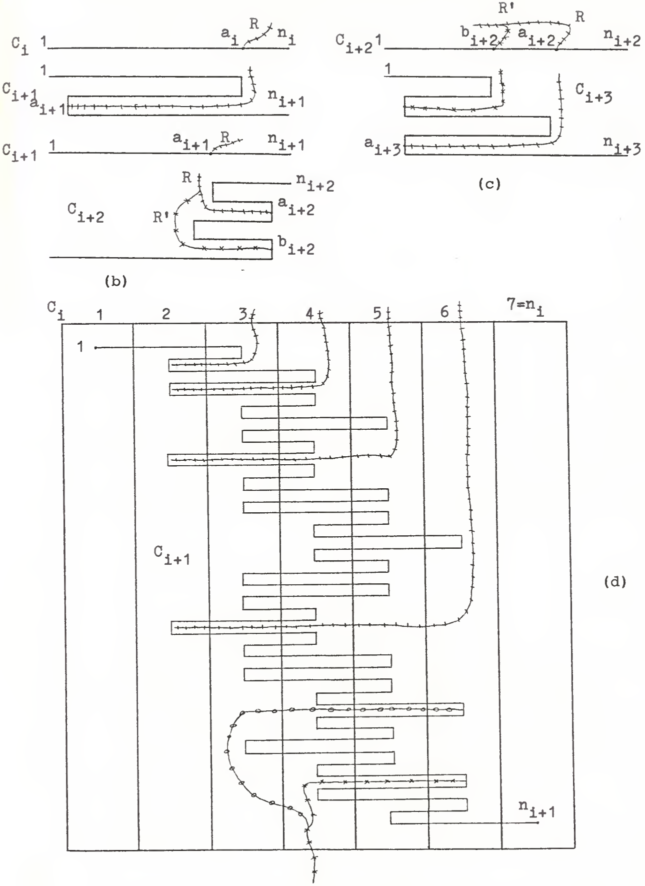
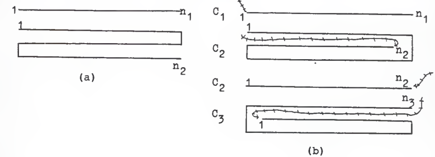

C_i
1
a_i
R
n_i
1
C_{i+1}
a_{i+1}
n_{i+1}
C_{i+1}
1
a_{i+1}
R
n_{i+1}
R
n_{i+2}
C_{i+2}
R'
a_{i+2}
b_{i+2}
(b)
R'
R
C_{i+2}
1
b_{i+2}
a_{i+2}
n_{i+2}
1
C_{i+3}
a_{i+3}
n_{i+3}
(c)
C_i
1
2
3
4
5
6
$7=n_i$
1
C_{i+1}
(d)
n_{i+1}

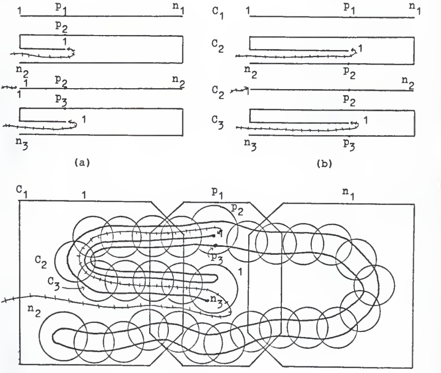
Figure 2.11



(a)

(b)

Figure 2.12

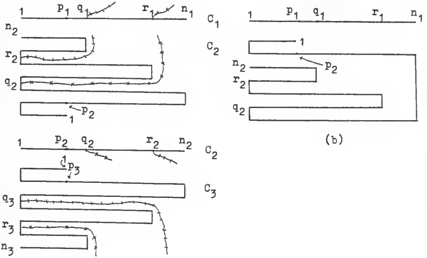


(a)

(b)

(c)

Figure 2.13



(a)

(b)

Figure 2.14

CHAPTER 3
PRINCIPAL EMBEDDINGS OF ATRIODIC PLANE CONTINUA

3.1 Introduction

Several known examples of tree like continua which admit fixed point free maps are atriodic and nonchainable, and each proper subcontinuum is an arc ([Bellamy, 1981] , [Oversteegen and Rogers, 1980]). In [1980a] Brechner and Mayer show that if there is an indecomposable nonseparating plane continuum which admits a fixed point free map, it must have a Lake-of-Wada channel in every embedding. This is a consequence of independent results of Bell [1978] and Sieklucki [1968] . In this chapter an example is given of a tree-like, atriodic, nonchainable, indecomposable nonseparating plane continuum each of whose proper subcontinua is an arc which has an embedding with a Lake-of-Wada channel. The construction of the example is based upon an example of Ingram's [1972] . Ingram's example is a tree-like, atriodic, nonchainable, nonseparating plane continuum each of whose proper subcontinua is an arc. However, since it has an embedding with no Lake-of-Wada channel, it has the fixed point property ([Brechner and Mayer, 1980a], Theorem 4.1). Both Ingram's example and ours are proved nonchainable by showing they have a positive span.

We construct our example as an inverse limit of X's, so will refer to it as the X-odic continuum. The resulting continuum, X, is homemorphic to a continuum defined as the intersection of a defining sequence (in the sense of Ingram and Cook [1967]) of tree-covers, and we make use of both constructions in our proofs. We were informed by C. Hagopian that he has a similar example as an inverse limit of X's, but with exactly two Lake-of-Wada channels.

In Section 3.2 we construct the example, X, and prove that it is atriodic and nonchainable. The main theorems of this section are 3.2.4 showing that X is atriodic, and 3.2.7 showing that X is nonchainable. The most complex theorem is 3.2.6 showing that X has properties sufficient to guarantee positive span.

In Section 3.3 we show that X has an embedding in the plane with a Lake-of-Wada channel. We also raise some questions about X and the fixed point problem for nonseparating plane continua.

All spaces are metric and distance functions are as usual for spaces and their products. All functions (maps) are continuous.

3.2 The X-odic Continuum.

We shall define the X-odic continuum X in two ways. It is evident that the continua so defined are homeomorphic. The second definition will be such that $X \subset E^2$, and so provides an embedding of X in the plane.

3.2.1 Inverse Limit Definition of X.

Let X_1 be the union of intervals $[-1,1]$ on the coordinate axes in the xy-plane. For convenience we designate $(0,1)$ as A, $(-1,0)$ as B, $(1,0)$ as C, $(0,-1)$ as D, and $(0,0)$ as O. X_1 is then the identification of four intervals, [OA] , [OB] , [OC] , [OD] , at a single point O. By $\frac{pA}{q}$ mean the point $(0,\frac{p}{q})$ and similarly for B, C, and D. Thus $[O\frac{pC}{q})]$ denotes the interval on the x-axis from $(0,0)$ to $(\frac{p}{q},0)$.

Let $f : X_1 \to X_1$ be a map carrying

$O\frac{A}{6}$ onto OB, OR (order-reversing and proportionally)

$\frac{A}{6}\,\frac{A}{3}$ onto OA, OP (order-preserving and proportionally)

$\frac{A}{3}\,\frac{A}{2}$ onto OA, OR

$\frac{A}{2}\,\frac{2A}{3}$ onto OC, OP

$\frac{2A}{3}\,\frac{5A}{6}$ onto OC, OR

$\frac{5A}{6}\,A$ onto OD, OP

$O\frac{B}{3}$ onto OB, OR

$\frac{B}{3}\,\frac{B}{2}$ onto $O\frac{A}{3}$, OP

$\frac{B}{2}\,\frac{2B}{3}$ onto $O\frac{A}{3}$, OR

$\frac{2B}{3}B$ onto OC, OP

$O\frac{C}{2}$ onto OB, OR

$\frac{C}{2}C$ onto OD, OP

$O\frac{D}{2}$ onto OB, OR

$\frac{D}{2}D$ onto OC, OP

A schematic diagram of f is given in Figure 3.1.

For each i, let $X_i = X_1$ and $f_i = f$. We define X by:

$$X = \varprojlim \{ X_i, f_i \}$$

Let $f_1^n : X_n \to X_1$ be defined by

$$f_1^n = f_1 \circ f_2 \circ f_3 \circ \ . \ . \ . \circ f_n$$

3.2.2 Defining Sequence for X.

For each n, let T_n be a collection of open disks in the plane, E^2 such that

(1) T_{n+1} strongly refines T_n; that is, for each L in T_{n+1} there is some $M \subset T_n$ such that $\bar{L} \subset M$.

(2) The mesh of T_n is less than $1/2^n$.

(3) T_n is a coherent collection of four subchains with exactly one junction link designated as follows:

(a) The junction link is O_n.

(b) The four endlinks are A_n, B_n, C_n, and D_n.

(c) The four subchains are O_nA_n, O_nB_n, O_nC_n, and O_nD_n.

(4) T_{n+1} follows in T_n the pattern suggested by function f, including orientation with respect to the plane. (That is, if T_n were "straightened out" by some orientation-preserving homemorphism of E^2, then T_{n+1} would sit in T_n exactly as Figure 3.1 suggests.)

Then $X = \bigcap_{n=1}^{\infty} T_n$.

Figure 3.2 illustrates the first three stages in the construction of X. The third stage is represented by its nerve. Since T_{n+1} follows the pattern of f in T_n, we may conveniently refer to links of T_n on analogy with points of X_n. That is, $A_{n+1}/3$ is a link of chain $O_{n+1}A_{n+1}$ that corresponds to point $(0,\frac{A}{3})$ of X_n. Note $A_{n+1}/3$ is a link of T_{n+1} sitting in link A_n of T_n. Similarly, link $B_{n+1}/2$ of T_{n+1} is a subset of link $A_n/3$ of T_n. We could have defined the pattern that T_{n+1} follows in T_n directly in terms of such intermediate links and the chains between them. We require our chains have as few bends as possible. (Too many bends and we might get pseudo-arcs as subcontinua.) We will call each T_n (which is a tree cover of X) an X-od cover of X. An open cover of X by subsets of X is derived from the above by letting T_n' be the collection of open sets of X such that

$$L' \text{ is in } T_n' \text{ iff } L' = L \cap X \text{ for some } L \text{ is in } T_n$$

It is clear from our definition that $X \subset E^2$ in the construction of 3.2.2. We will refer to this embedding of X in what follows.

3.2.3 Definition of the Span of a Function.

The span, σf, of function $f: X \to Y$ is the least upper bound of all numbers ε for which there is a connected subset Z_ε of XxX such that $\pi_1(Z_\varepsilon) = \pi_2(Z_\varepsilon)$ and $d(f(x), f(y)) \geq \varepsilon$ for all (x,y) in Z_ε. The span, σX, of space X is the span of the identity function on X (Ingram [1972] , due to Lelek [1968]).

3.2.4 Theorem

Let X be the continuum defined by X-od tree covers in 3.2.2. Then X is atriodic.

Proof. We will show every proper subcontinuum of X is chainable. That X is then atriodic follows from Theorem 3 of Ingram [1968].

Let H be a proper subcontinuum of X. Let $\{T_n\}_{n=1}^{\infty}$ be the defining sequence of tree-covers for X, defined in 3.2.2. For each n, let F_n be that subcollection of T_n that covers H minimally. Then $H = \bigcap_{n=1}^{\infty} F_n$. For some N, for all $k \geq N$, $F_k \neq T_k$, for otherwise H = M. Since H is a continuum, each F_n must be coherent. We consider two cases:

If for some subsequence $\{F_j\}_{j=1}^{\infty}$ of $\{F_n\}_{n=1}^{\infty}$, F_j does not include O_j, then $H = \bigcap_{j=1}^{\infty} F_j$ is chainable as each F_j is a chain.

So assume there is no such subsequence. Then for some J, for all $k \geq J$, O_k is in F_k. Let k be chosen as the maximum of N, J. Then O_{k+3} is in F_{k+3}. Now $O_{k+3} \subset B_{k+2}$ in F_{k+2}. But O_{k+2} is in F_{k+2}. Since F_{k+2} is coherent, chain $O_{k+2}B_{k+2} \subset F_{k+2}$. Observe that

$$O_{k+2}B_{k+2} \subset O_{k+1}B_{k+1} \cup O_{k+1}\ A_{k+1}/3 \cup O_{k+1}C_{k+1}$$

$$O_{k+1}B_{k+1} \subset O_kB_k \ \cup \ O_{k+1}A_{k+1}/3 \ \cup \ O_kC_k,$$

$$O_{k+1}A_{k+1}/3 \subset O_kB_k \ \cup \ O_kA_k,$$

$$O_{k+1}C_{k+1} \subset O_kB_k \ \cup \ O_kD_k$$

Hence, $O_{k+2}B_{k+2}$ meets every element of T_k. Since $O_{k+2}B_{k+2} \subset F_{k+2}$ and F_k contains F_{k+2}, $F_k = T_k$. But $k \geq N$, so $F_k \neq T_k$, a contradiction. Therefore, H is chainable, and by Ingram's theorem noted above, atriodic. QED

Figure 3.3 shows how O_3B_3 sits in T_1.

To show that X is nonchainable we will use Theorem 4 of Ingram [1972].

3.2.5 Theorem.

If $X = \varprojlim \{X_i, f_i\}$ with each X_i compact, and for $\varepsilon > 0$, $\sigma f_1^n \geq \varepsilon$, for each n, then $\sigma X > 0$.

Lelek [1968] observes that $\sigma X > 0$ implies X is nonchainable.

To apply 3.2.5 we require the following as a lemma. The proof is an application of the method of proof of Theorem 2 in Ingram [1972].

3.2.6 Theorem.

There exists a sequence $Z_1, Z_2 \ldots$ of subcontinua of $X_1 x X_1$ such that for each n, $\pi_1(Z_n) = \pi_2(Z_n) = X_1$, $fxf(Z_{n+1}) = Z_n$, $Z_n = Z_n^{-1}$, and if (p,q) is in Z_1, then $d(p,q) \geq \frac{1}{3}$, for all n.

Proof. Our proof is by induction on n. Z_1 is the union of the following twenty subcontinua of $X_1 \times X_1$:

$$m_1=([OB]\times\{A\})\cup(\{B\}\times[OA])\quad m_2=([OA]\times\{B\})\cup(\{A\}\times[OB])$$

$$m_3=([OB]\times\{\tfrac{A}{3}\})\cup(\{B\}\times[O\tfrac{A}{3}])\quad m_4=([O\tfrac{A}{3}]\times\{B\})\cup(\{\tfrac{A}{3}\}\times[OB])$$

$$m_5=([OC]\times\{A\})\cup(\{C\}\times[OA])\quad m_6=([OA]\times\{C\})\cup(\{A\}\times[OC])$$

$$m_7=([OC]\times\{\tfrac{A}{3}\})\cup(\{C\}\times[O\tfrac{A}{3}])\quad m_8=([O\tfrac{A}{3}]\times\{C\})\cup(\{\tfrac{A}{3}\}\times[OC])$$

$$m_9=([OD]\times\{A\})\cup(\{D\}\times[OA])\quad m_{10}=([OA]\times\{D\})\cup(\{A\}\times[OD])$$

$$m_{11}=([OD]\times\{\tfrac{A}{3}\})\cup(\{D\}\times[O\tfrac{A}{3}])\quad m_{12}=([O\tfrac{A}{3}]\times\{D\})\cup(\{\tfrac{A}{3}\}\times[OD])$$

$$m_{13}=([OC]\times\{B\})\cup(\{C\}\times[OB])\quad m_{14}=([OB]\times\{C\})\cup(\{B\}\times[OC])$$

$$m_{15}=([OD]\times\{B\})\cup(\{D\}\times[OB])\quad m_{16}=([OB]\times\{D\})\cup(\{B\}\times[OD])$$

$$m_{17}=([OD]\times\{C\})\cup(\{D\}\times[OC])\quad m_{18}=([OC]\times\{D\})\cup(\{C\}\times[OD])$$

$$m_{19}=([O\tfrac{A}{3}]\times\{A\})\cup(\{O\}\times[\tfrac{5A}{6}A])\quad m_{20}=([\tfrac{5A}{6}A]\times\{O\})\cup(\{A\}\times[O\tfrac{A}{3}])$$

Each of m_i $(1 \leq i \leq 20)$ is a continuum. We designate k_j $(1 \leq j \leq 10)$ below as unions of certain m_i containing a common point; hence each k_j is a continuum. Figures 5 and 6 illustrate Z_1.

$k_1 = m_1 \cup m_5 \cup m_9 \cup m_{19}$ (O,A) is in $m_1 \cap m_5 \cap m_9 \cap m_{19}$

$k_2 = m_2 \cup m_6 \cup m_{10} \cup m_{20}$ (A,O) is in $m_2 \cap m_6 \cap m_{10} \cap m_{20}$

$k_3 = m_2 \cup m_4 \cup m_{13} \cup m_{15}$ (O,B) is in $m_2 \cap m_4 \cap m_{13} \cap m_{15}$

$k_4 = m_1 \cup m_3 \cup m_{14} \cup m_{16}$ (B,O) is in $m_1 \cap m_3 \cap m_{14} \cap m_{16}$

$k_5 = m_6 \cup m_8 \cup m_{14} \cup m_{17}$ (O,C) is in $m_6 \cap m_8 \cap m_{14} \cap m_{17}$

$k_6 = m_5 \cup m_7 \cup m_{13} \cup m_{18}$ (C,O) is in $m_5 \cap m_7 \cap m_{13} \cap m_{18}$

$k_7 = m_{10} \cup m_{12} \cup m_{16} \cup m_{18}$ (O,D) is in $m_{10} \cap m_{12} \cap m_{16} \cap m_{18}$

$k_8 = m_9 \cup m_{11} \cup m_{15} \cup m_{17}$ (D,O) is in $m_9 \cap m_{11} \cap m_{15} \cap m_{17}$

$k_9 = m_3 \cup m_7 \cup m_{11}$ $(0,\frac{A}{3})$ is in $m_3 \cap m_7 \cap m_{11}$

$k_{10} = m_4 \cup m_8 \cup m_{12}$ $(\frac{A}{3},0)$ is in $m_4 \cap m_8 \cap m_{12}$

Observe that $m_1 \subset k_1 \cap k_4$, $m_{16} \subset k_4 \cap k_7$, $m_{10} \subset k_7 \cap k_2$, $m_6 \subset k_2 \cap k_5$, $m_{17} \subset k_5 \cap k_8$, $m_{15} \subset k_8 \cap k_3$, $m_{13} \subset k_3 \cap k_6$. Hence, $\bigcup_{j=1}^{8} k_j$ is a continuum and further $(k_9 \cup k_{10}) \subset \bigcup_{j=1}^{8} k_j = \bigcup_{i=1}^{20} m_i$. Hence $Z_1 = \bigcup_{i=1}^{20} m_i$ is a continuum. We further observe that

For each odd i, $(1 \leq i \leq 19)$, $m_i^{-1} = m_{i+1}$ so $Z_1^{-1} = Z_1$,

$\pi_1(m_1 \cup m_{10} \cup m_{13} \cup m_{17}) = \pi_2(m_1 \cup m_{10} \cup m_{13} \cup m_{17}) = X_1$,

(B,C) is in m_{14}, (C,B) is in m_{13}, (B,A) is in m_1, (A,B) is in m_2, (B,D) is in m_{16}, (D,B) is in m_{15}

If (p,q) is in Z_1, then $d(p,q) \geq \frac{1}{3}$.

We adopt the convention that $< t,u >$ denotes a continuum M such that $\pi_1(M) = t$ and $\pi_2(M) = u$ and $< t,u >^{-1} = <u,t>$. Note that Z_1 is the union of 20 such continua in the same order as in the induction hypothesis below. If $< t,u >$ is a subcontinuum of Z_n, and v,w are subarcs of X_1 such that $f|v$ maps onto t and $f|_w$ maps onto u, both homemorphically, then

$L = (f^{-1}|v \times f^{-1}|w)(< t, u >)$ is a continuum such that

$\pi_1(L) = v$ and $\pi_2(L) = w$.

We denote this continuum as $L(<t,u>, v,w)$, or more briefly as L, and call it the lifting of $< t,u >$ with respect to v and w as defined in Ingram [1971]).

Induction hypothesis. Z_n is a continuum of $X_1 x X_1$ such that

(a) $\pi_1(Z_n) = \pi_2(Z_n) = X$.

(b) Z_n is the union of twenty continue denoted by

$$\langle OB,OA\rangle \quad \langle OA,OB\rangle \quad \langle OB,O\tfrac{A}{3}\rangle \quad \langle O\tfrac{A}{3},OB\rangle$$

$$\langle OC,OA\rangle \quad \langle OA,OC\rangle \quad \langle OC,O\tfrac{A}{3}\rangle \quad \langle O\tfrac{A}{3},OC\rangle$$

$$\langle OD,OA\rangle \quad \langle OA,OD\rangle \quad \langle OD,O\tfrac{A}{3}\rangle \quad \langle O\tfrac{A}{3},OD\rangle$$

$$\langle OC,OB\rangle \quad \langle OB,OC\rangle \quad \langle OD,OB\rangle \quad \langle OB,OD\rangle$$

$$\langle OD,OC\rangle \quad \langle OC,OD\rangle \quad \langle O\tfrac{A}{3},\tfrac{5A}{6}A\rangle \quad \langle \tfrac{5A}{6}A,O\tfrac{A}{3}\rangle$$

where $\langle t,u\rangle$ is a continuum M such that $\pi_1(M) = t$ and $\pi_2(M) = u$.

c) $\langle t,u\rangle^{-1} = \langle u,t\rangle$

d) There are five points in X_1, Z_n as follows:

x_1 is in $\frac{5A}{6}A$, with (x_1,O) in $\langle OA,OB\rangle \cap \langle OA,OC\rangle \cap \langle OA,OD\rangle \cap \langle \frac{5A}{6}A,O\frac{A}{3}\rangle$

and (O,x_1) in $\langle OB,OA\rangle \cap \langle OC,OA\rangle \cap \langle OD,OA\rangle \cap \langle O\frac{A}{3},\frac{5A}{6}A\rangle$

x_2 is in $\frac{2B}{3}B$, with (x_2,O) in $\langle OB,OA\rangle \cap \langle OB,OC\rangle \cap \langle OB,OD\rangle \cap \langle OB,O\frac{A}{3}\rangle$

and (O,x_2) in $\langle OA,OB\rangle \cap \langle OC,OB\rangle \cap \langle OD,OB\rangle \cap \langle O\frac{A}{3},OB\rangle$

x_3 is in $\frac{C}{2}C$, with (x_3,O) in $\langle OC,OA\rangle \cap \langle OC,OB\rangle \cap \langle OC,OD\rangle \cap \langle OC,O\frac{A}{3}\rangle$

and (O,x_3) in $\langle OA,OC\rangle \cap \langle OB,OC\rangle \cap \langle OD,OC\rangle \cap \langle O\frac{A}{3},OC\rangle$

x_4 is in $\frac{D}{2}D$, with (x_4,O) in $\langle OD,OA\rangle \cap \langle OD,OB\rangle \cap \langle OD,OC\rangle \cap \langle OD,O\frac{A}{3}\rangle$

and (O,X_4) in $\langle OA,OD\rangle \cap \langle OB,OD\rangle \cap \langle OC,OD\rangle \cap \langle O\frac{A}{3},OD\rangle$

x_5 is in $\frac{A}{6}\,\frac{A}{3}$, with (x_5,O) in $\langle O\frac{A}{3},OB\rangle \cap \langle O\frac{A}{3},OC\rangle \cap \langle O\frac{A}{3},OD\rangle$

and (O,x_5) in $\langle OB,O\frac{A}{3}\rangle \cap \langle OC,O\frac{A}{3}\rangle \cap \langle OD,O\frac{A}{3}\rangle$

e) There are three points in X_1, Z_n as follows:

z_1 is in OC with (B,z_1) in $\langle OB,OC\rangle$ and (z_1,B) in $\langle OC,OB\rangle$

z_2 is in OA with (B,z_2) in $\langle OB,OA\langle$ and (z_2,B) in $\langle OA,OB\rangle$

z_3 is in OD with (B,z_3) in $\langle OB,OD\langle$ and (z_3,B) in $\langle OD,OB\rangle$

Base Case. Observe that Z_1 meets all of the above conditions.

Induction. Our induction will be to construct Z_{n+1} by lifting Z_n, so Z_{n+1} satisfies (a)-(e) and $fxf(z_{n+1}) = Z_n$. By a_i $(1 \leq i \leq 20)$ we will denote the twenty continua whose union is Z_{n+1}. Furthermore, $\langle t,u \rangle'$ will denote a continuum which "corresponds" to $\langle t,u \rangle$ of Z_n. This proof closely follows Ingram's method in [1972], as does the inductive hypothesis above.

Continua a_1, a_2, a_3, and a_4.

$$a_1 = \langle OB,OA\rangle' = L_1^1(\langle OC,OB\rangle, \tfrac{2B}{3}\,B, O\tfrac{A}{6}) \cup L_2^1(\langle OC,OA\rangle, \tfrac{2B}{3}B, \tfrac{A}{6}\,\tfrac{A}{3}) \cup$$

$$L_3^1(\langle O\tfrac{A}{3}, \tfrac{5A}{6}\,A\rangle, \tfrac{B}{2}\,\tfrac{2B}{3}, \tfrac{11AA}{363}) \cup L_4^1(\langle O\tfrac{A}{3}\,\tfrac{5A}{6}A\rangle, \tfrac{BB}{32}, \tfrac{11AA}{36\ 3}) \cup$$

$$L_5^1(\langle OB,OA\rangle, O\tfrac{B}{3}, \tfrac{AA}{63}) \cup L_6^1(\langle OB,OA\rangle, \tfrac{OB}{3}, \tfrac{AA}{32}) \cup$$

$$L_7^1(\langle OB, OC\rangle, O\frac{B}{3}, \frac{A}{2}\frac{2A}{3}) \cup L_8^1(\langle OB, OC\rangle, \frac{OB}{3}, \frac{2A}{3}\ \frac{5A}{6}) \cup$$

$$L_9^1(\langle OB, OD\rangle, O\frac{B}{3}, \frac{5A}{6}A)$$

$a_2 = \langle OA, OB\rangle' = a_1^{-1}$

That a_1 is a continuum follows from

(x_3, O) is in $\langle OC, OB\rangle \cap \langle OC, OA\rangle$,

so $(f^{-1}|\frac{2B}{3}B(x_3), \frac{A}{6})$ is in $L_1^1 \cap L_2^1$

(O, x_1) is in $\langle OC, OA\rangle \cap \langle O\frac{A}{3}, \frac{5A}{6}A\rangle$,

so $(\frac{2B}{3}, f^{-1}|\frac{11A}{36}\ \frac{A}{3}\ (x_1))$ is in $L_2^1 \cap L_3^1$

There is a y in $\frac{5A}{6}$, with $(\frac{A}{3}, y)$ in $\langle O\frac{A}{3}, \frac{5A}{6}A\rangle$,

so $(\frac{B}{2}, f^{-1}|\frac{11A}{36}\ \frac{A}{3}(y))$ is in $L_3^1 \cap L_4^1$

(O, x_1) is in $\langle O\frac{A}{3}, \frac{5A}{6}A\rangle \cap \langle OB, OA\rangle$,

so $(\frac{B}{3}, f^{-1}|\frac{11A}{36}\ \frac{A}{3}\ (x_1))$ is in $L_4^1 \cap L_5^1$

There is a y in OB, with (y, A) in $\langle OB, OA\rangle$,

so $(f^{-1}|O\frac{B}{3}(y), \frac{A}{3})$ is in $L_5^1 \cap L_6^1$

(x_2,O) is in $<OB,OA> \cap <OB,OC>$,

$$\text{so } (f^{-1}|O\tfrac{B}{3}(x_2,\tfrac{A}{2}) \text{ is in } L_6^1 \cap L_7^1$$

There is a y in OB with (y,C) in $<OB,OC>$,

$$\text{so } (f^{-1}|O\tfrac{B}{3}\ (y),\tfrac{2A}{3}) \text{ is in } L_7^1 \cap L_8^1$$

(x_2,O) is in $<OB,OC> \cap <OB,OD>$,

$$\text{so } (f^{-1}|O\tfrac{B}{3}\ (x_2),\ \tfrac{5A}{6}\ ) \text{ is in } L_8^1 \cap L_9^1$$

So a_2 is also a continuum. Also

$$\pi_1(a_1) = OB \ \& \ \pi_2(a_1) = OC \ \& \ fxf\ (a_1) \subset Z_n$$

$$\pi_1(a_2) = OC \ \& \ \pi_2(a_2) = OB \ \& \ fxf\ (a_2) \subset Z_n$$

$a_3 = <OB,O\tfrac{A}{3}>' = \bigcup_{i=1}^{5} L_1^3$ where $L_i^3 = L_i^1$ for $1 \leq i \leq 5$.

$a_4 = <O\tfrac{A}{3},OB>' = a_3^{-1}$

The proof that $a_1(a_2)$ is a continuum contains the proof that $a_3(a_4)$ is a continuum. Further

$$\pi_1(a_3) = OB \ \& \ \pi_2(a_3) = O\tfrac{A}{3} \ \& \ fxf\ (a_3) \subset Z_n$$
$$\pi_1(a_4) = O\tfrac{A}{3} \ \& \ \pi_2(a_4) = OB \ \& \ fxf\ (a_4) \subset Z_n$$

Continua a_5, a_6, a_7, and a_8.

$$a_5 = \langle OC,OA\rangle' = \quad L_1^5(\langle OD,OB\rangle,\frac{C}{2}C,O\frac{A}{6}) \cup L_2^5(\langle OD,OA\rangle,\frac{C}{2}C,\frac{AA}{63}) \cup$$

$$L_3^5(\langle OB,OA\rangle,O\frac{C}{2},\frac{AA}{63}) \cup L_4^5(\langle OB,OA\rangle,O\frac{C}{2},\frac{AA}{32}) \cup$$

$$L_5^5(\langle OB,OC\rangle,O\frac{C}{2},\frac{A}{2}\ \frac{2A}{2}) \cup L_6^5(\langle OB,OC\rangle,O\frac{C}{2},\frac{2A}{3}\ \frac{5A}{6}) \cup$$

$$L_7^5(\langle OB,OD\rangle,O\frac{C}{2},\frac{5A}{6}A)$$

$a_6 = \langle OA,OC\rangle' = a_5^{-1}$

That a_5 is a continuum follows from

(x_4,O) is in $\langle OD,OB\rangle \cap \langle OD,OA\rangle$,

so $(f^{-1}|\frac{C}{2}C(x_4),\frac{A}{6})$ is in $L_1^5 \cap L_2^5$

(O,x_1) is in $\langle OD,OA\rangle \cap \langle OB,OA\rangle$,

so $(\frac{C}{2},f^{-1}|\frac{AA}{63}(x_1))$ is in $L_2^5 \cap L_3^5$

There is a y in OB with (y,A) in $\langle OB,OA\rangle$,

so $(f^{-1}|O\frac{C}{2}(y),\frac{A}{3})$ is in $L_3^5 \cap L_4^5$

(x_2,O) is in $\langle OB,OA\rangle \cap \langle OB,OC\rangle$,

so $(f-^{1}|O\frac{C}{2}(x_2),\frac{A}{2})$ is in $L_4^5 \cap L_5^5$

There is a y in OB with (y,C) in $\langle OB,OC\rangle$,

$$\text{so } (f^{-1}|O\tfrac{C}{2}(y), \tfrac{2A}{3}) \text{ is in } L_5^5 \cap L_6^5$$

(x_2,O) is in $\langle OB,OC\rangle \cap \langle OB,OD\rangle$,

$$\text{so } (f^{-1}|O\tfrac{C}{2}(x_2), \tfrac{5A}{6}) \text{ is in } L_6^5 \cap L_7^5$$

So a_6 is also a continuum. Also

$$\pi_1(a_5) = OC \;\&\; \pi_2(a_5) = OA \;\&\; fxf\,(a_5) \subset Z_n$$
$$\pi_1(a_6) = OA \;\&\; \pi_2(a_6) = OC \;\&\; fxf\,(a_6) \subset Z_n$$

$a_7 = \langle OC, O\frac{A}{3}\rangle' = \bigcup_{i=1}^{3} L_i^7$ where $L_i^7 = L_i^5$ for $1 \leq i \leq 3$.
$a_8 = \langle O\frac{A}{3}, OC\rangle' = a_7^{-1}$

That $a_7(a_8)$ is a continuum follows from the proof for $a_5(a_6)$. Further

$$\pi_1(a_7) = OC \;\&\; \pi_2(a_7) = O\tfrac{A}{3} \;\&\; fxf\,(a_7) \subset Z_n$$

$$\pi_1(a_8) = O\tfrac{A}{3} \;\&\; \pi_2(a_8) = OC \;\&\; fxf\,(a_8) \subset Z_n$$

Continua a_9, a_{10}, a_{11}, and a_{12}.

$$a_9 = \langle OD,OA\rangle' = L_1^9(\langle OC,OB\rangle,\tfrac{D}{2}D,O\tfrac{A}{6}) \cup L_2^9(\langle OC,OA\rangle,\tfrac{D}{2}D\tfrac{AA}{63}) \cup$$

$$L_3^9(\langle OB,OA\rangle,O\tfrac{D}{2},\tfrac{AA}{63}) \cup L_4^9(\langle OB,OA\rangle,O\tfrac{D}{2},\tfrac{AA}{32}) \cup$$

$$L_5^9(\langle OB,OC\rangle,O\tfrac{D}{2},\tfrac{A\,2A}{2\;\;3}) \cup L_6^9(\langle OB,OC\rangle,O\tfrac{D}{2},\tfrac{2A\,5A}{3\;\;\;6}) \cup$$

$$L_7^9(\langle OB,OD\rangle,O\tfrac{D}{2},\tfrac{5A}{6}A)$$

$a_{10} = \langle OA,OD\rangle' = a_9^{-1}$

That a_9 is a continuum follows from

$(x_3,0)$ is in $\langle OC,OB\rangle \cap \langle OC,OA\rangle$,

so $(f^{-1}|\tfrac{D}{2}D(x_3),\tfrac{A}{6})$ in $L_1^9 \cap L_2^9$

$(0,x_1)$ is in $\langle OC,OA\rangle \cap \langle OB,OA\rangle$,

so $(\tfrac{D}{2},f^{-1}|\tfrac{AA}{63}(x_1))$ in $L_2^9 \cap L_3^9$

There is a y in OB with (y,A) in $\langle OB,OA\rangle$,

so $(f^{-1}|O\tfrac{D}{2}(y),\tfrac{A}{3})$ in $L_3^9 \cap L_4^9$

$(x_2,0)$ is in $\langle OB,OA\rangle \cap \langle OB,OC\rangle$,

so $(f^{-1}|O\tfrac{D}{2}(x_2),\tfrac{A}{2})$ in $L_4^9 \cap L_5^9$

There is a y in OB with (y,C) in $\langle OB,OC\rangle$,

$$\text{so } (f^{-1}|O\tfrac{D}{2}(y), \tfrac{2A}{3}) \text{ is in } L_5^9 \cap L_6^9$$

$(x_2,0)$ is in $\langle OB,OC\rangle \cap \langle OB,OD\rangle$,

$$\text{so } (f^{-1}|O\tfrac{D}{2}(x_2), \tfrac{5A}{6}) \text{ is in } L_6^9 \cap L_7^9$$

So a_{10} is also a continuum. Also

$$\pi_1(a_9) = OD \ \& \ \pi_2(a_9) = OA \ \& \ f{\times}f\ (a_9) \subset Z_n$$

$$\pi_1(a_{10}) = OA \ \& \ \pi_2(a_{10}) = OD \ \& \ f{\times}f\ (a_{10}) \subset Z_n$$

$$a_{11} = \langle OD, O\tfrac{A}{3}\rangle' = \bigcup_{i=1}^{3} L_i^{11} \text{ where } L_i^{11} = L_i^9 \text{ for } 1 \leq i \leq 3.$$

$$a_{12} = \langle O\tfrac{A}{3}, OD\rangle' = a_{11}^{-1}$$

That $a_{11}(a_{12})$ is a continuum follows from proof for $a_9(a_{10})$. Also

$$\pi_1(a_{11}) = OD \ \& \ \pi_2(a_{11}) = O\tfrac{A}{3} \ \& \ f{\times}f\ (a_{11}) \subset Z_n$$

$$\pi_1(a_{12}) = O\tfrac{A}{3} \ \& \ \pi_2(a_{12}) = OD \ \& \ f{\times}f\ (a_{12}) \subset Z_n$$

Continua a_{13} and a_{14}

$$a_{13} = \langle OC,OB\rangle' = L^{13}_{1}(\langle OD,OB\rangle,\tfrac{C}{2}C,O\tfrac{B}{3}) \cup L^{13}_{2}\;(\langle OD,O\tfrac{A}{3}\rangle,\tfrac{C}{2}C,\tfrac{BB}{32}) \cup$$

$$L^{13}_{3}(\langle OB,O\tfrac{A}{3}\rangle,O\tfrac{C}{2},\tfrac{BB}{32}) \cup L^{13}_{4}\;(\langle OB,O\tfrac{A}{3}\rangle,O\tfrac{C}{2},\tfrac{B\,2B}{2\;\;3}) \cup$$

$$L^{13}_{5}(\langle OB,OC\rangle,O\tfrac{C}{2},\tfrac{2B}{3}B)$$

$a_{14} = \langle OB,OC\rangle' = a^{-1}_{13}$

That a_{13} is a continuum follows from

(x_4,O) is in $\langle OD,OB\rangle \cap \langle OD,O\frac{A}{3}\rangle$,
so $(f^{-1}|\frac{C}{2}C(x_4),\frac{B}{3})$ is in $L^{13}_{1} \cap L^{13}_{2}$

(O,x_5) is in $\langle OD,O\frac{A}{3}\rangle \cap \langle OB,O\frac{A}{3}\rangle$,
so $(\frac{C}{2},f^{-1}|\frac{BB}{32}(x_5))$ is in $L^{13}_{2} \cap L^{13}_{3}$

There is a y in OB with $(y,\frac{A}{3})$ in $\langle OB,O\frac{A}{3}\rangle$,
so $(f^{-1}|O\frac{C}{2}(y),\frac{B}{2})$ is in $L^{13}_{3} \cap L^{13}_{4}$

(x_2,O) is in $\langle OB,O\frac{A}{3}\rangle \cap \langle OB,OC\rangle$,
so $(f^{-1}|O\frac{C}{2}(x_2),\frac{2B}{3}) \in L^{13}_{4} \cap L^{13}_{5}$

So a_{14} is also a continuum. Also

$$\pi_1(a_{13}) = OC \;\&\; \pi_2(a_{13}) = OB \;\&\; fxf\,(a_{13}) \subset Z_n$$
$$\pi_1(a_{14}) = OB \;\&\; \pi_2(a_{14}) = OC \;\&\; fxf\,(a_{14}) \subset Z_n$$

Continua a_{15} and a_{16}.

$$a_{15} = \langle OD,OB\rangle' = L^{15}_{1}(\langle OC,OB\rangle,\tfrac{D}{2}D,O\tfrac{B}{3}) \cup L^{15}_{2}(\langle OC,O\tfrac{A}{3}\rangle,\tfrac{D}{2}D,\tfrac{BB}{32}) \cup$$

$$L^{15}_{3}(\langle OB,O\tfrac{A}{3}\rangle,O\tfrac{D}{2},\tfrac{BB}{32}) \cup L^{15}_{4}(\langle OB,O\tfrac{A}{3}\rangle,O\tfrac{D}{2},\tfrac{B}{2}\ \tfrac{2B}{3}) \cup$$

$$L^{15}_{5}(\langle OB,OC\rangle,O\tfrac{D}{2},\tfrac{2B}{3}B)$$

$$a_{16} = \langle OB,OD\rangle' = a^{-1}_{15}$$

That a_{15} is a continuum follows from

$(x_3,0)$ is in $\langle OC,OB\rangle \cap \langle OC,O\tfrac{A}{3}\rangle$,
so $(f^{-1}|\tfrac{D}{2}D(x_3),\ \tfrac{B}{3})$ is in $L^{15}_1 \cap L^{15}_2$

$(0,x_5)$ is in $\langle OC,O\tfrac{A}{3}\rangle \cap \langle OB,O\tfrac{A}{3}\rangle$,
so $(\tfrac{D}{2},f^{-1}|\tfrac{B}{3}\ \tfrac{B}{2}(x_5))$ is in $L^{15}_2 \cap L^{15}_3$

There is a y in OB with $(y,\tfrac{A}{3})$ in $\langle OB,O\tfrac{A}{3}\rangle$,
so $(f^{-1}|O\tfrac{D}{2}(y),\tfrac{B}{2})$ is in $L^{15}_3 \cap L^{15}_4$

$(x_2,0)$ is in $\langle OB,O\tfrac{A}{3}\rangle \cap \langle OB,OC\rangle$,
so $(f^{-1}|O\tfrac{D}{2}(x_2),\tfrac{2B}{3})$ is in $L^{15}_4 \cap L^{15}_5$

So a_{16} is also a continuum. Also

$$\pi_1(a_{15}) = OD \ \&\ \pi_2(a_{15}) = OB \ \&\ f{\times}f\ (a_{15}) \subset Z_n$$

$$\pi_1(a_{16}) = OB \ \&\ \pi_2(a_{16}) = OD \ \&\ f{\times}f\ (a_{16}) \subset Z_n$$

Continua a_{17} and a_{18}.

$$a_{17} = \langle OD,OC\rangle' = L_1^{17}(\langle OC,OB\rangle,\tfrac{D}{2}D,O\tfrac{C}{2}) \cup L_2^{17}(\langle OC,OD\rangle,\tfrac{D}{2}D,\tfrac{C}{2}C) \cup$$

$$L^{17}_3(\langle OB,OD\rangle,O\tfrac{D}{2},\tfrac{C}{2}C)$$

$$a_{18} = \langle OC,OD\rangle' = a^{-1}_{17}$$

That a_{17} is a continuum follows from

(x_3,O) is in $\langle OC,OB\rangle \cap \langle OC,OD\rangle$,
so $(f^{-1}|\frac{D}{2}D(x_3),\frac{C}{2})$ is in $L^{17}_1 \cap L^{17}_2$

(O,x_4) is in $\langle OC,OD\rangle \cap \langle OB,OD\rangle$,
so $(\frac{D}{2},f^{-1}|\frac{C}{2}C(x_4))$ is in $L_2^{17} \cap L_3^{17}$

So a_{18} is also a continuum. Also

$$\pi_1(a_{17}) = OD \;\&\; \pi_2(a_{17}) = OC \;\&\; fxf\ (a_{17}) \subset Z_n$$

$$\pi_1(a_{18}) = OC \;\&\; \pi_2(a_{18}) = OD \;\&\; fxf\ (a_{18}) \subset Z_n$$

Continua a_{19} and a_{20}.

$$a_{19}=\langle O\tfrac{A}{3},\tfrac{5A}{6}A\rangle' = L^{19}_1(\langle OA,OD\rangle,\tfrac{A}{6}\ \tfrac{A}{3},\tfrac{5A}{6}A) \cup L^{19}_2(\langle OB,OC\rangle,O\tfrac{A}{6},\tfrac{5A}{6}A)$$

$$a_{20}= \langle\tfrac{5A}{6}A,O\tfrac{A}{3}\rangle' = a^{-1}_{19}$$

That a_{19} is a continuum follows from

(O,x_4) is in $\langle OA,OD\rangle \cap \langle OB,OD\rangle$,
so $(\frac{A}{6},f^{-1}\ |\frac{5A}{6}\ A(x_4))$ is in $L_1^{19} \cap L_2^{19}$

So a_{20} is also a continuum. Also

$$\pi_1(a_{19}) = O\frac{A}{3} \ \& \ \pi_2(a_{19}) = \frac{5A}{6}A \ \& \ f\times f\ (a_{19}) \subset Z_n$$

$$\pi_1(a_{20}) = \frac{5A}{6}A \ \& \ \pi_2(a_{20}) = O\frac{A}{3} \ \& \ f\times f\ (a_{20}) \subset Z_n$$

We now show $Z_{n+1} = \bigcup_{i=1}^{20} a_i$ is a continuum satisfying (a) -(e) of the induction hypothesis and such that fxf $(Z_{n+1}) = Z_n$. Designate the following five points:

$$x_1' = f^{-1}\ \frac{5A}{6}A(z_3) \ \& \ x_2' = f^{-1}\ \frac{2B}{3}B(Z_1)$$

$$x_3' = f^{-1}\ \frac{C}{2}C(z_3) \quad \& \ x_4' = f^{-1}\ \frac{D}{2}D(Z_1)$$

$$x_5' = f^{-1}\ \frac{AA}{63}(z_2)$$

These points will be shown below to satisfy (d) and will be used to show b_i $(1 \leq i \leq 10)$, defined below, are each continua. We explicitly define b_i for odd i, and note that for even i, $b_i = b_{i-1}^{-1}$.

$b_1 = a_2 \cup a_6 \cup a_{10} \cup a_{20}$, where L_9^2 implies $(x_1',0)$ is in a_2, L_7^6 implies $(x_1',0)$ is in a_6, L_7^{10} implies $(x_1',0)$ is in a_{10}, & L_2^{20} implies $(x_1',0)$ is in a_{20}.

$b_3 = a_1 \cup a_{14} \cup a_{16} \cup a_3$, where L_1^1 implies $(x_2',0)$ is in a_1, L_5^{14} implies $(x_2',0)$ is in a_{14}, L_5^{16} implies $(x_2',0)$ is in a_{16}, & L_1^3 implies $(x_2',0)$ is in a_3.

$b_5 = a_5 \cup a_{13} \cup a_{18} \cup a_7$, where L_1^5 implies $(x_3',0)$ is in a_5 , L_1^{13} implies $(x_3'0)$ is in a_{13}, L_3^{18} implies $(x_3',0)$ is in a_{18}, & L_1^7 implies $(x_3',0)$ is in a_7 .

$b_7 = a_9 \cup a_{15} \cup a_{17} \cup a_{11}$,where L_1^9 implies $(x_4',0)$ is in a_9, L_1^{15} implies $(x_4',0)$ is in a_{15}, L_1^{17} implies $(x_4',0)$ is in a_{17}, & L_1^{11} implies $(x_4',0)$ is in a_{11}.

$b_9 = a_4 \cup a_8 \cup a_{12}$, where L_5^4 implies $(x_5',0)$ is in a_4 , L_3^8 implies $(x_5',0)$ is in a_8, & L_3^2 implies $(x_5',0)$ is in a_{12}.

So x_1', x_2', x_3', x_4', x_5' satisfy (d). Observe that $a_2 \subset b_1 \cap b_4$, $a_{13} \subset b_4 \cap b_5$, $a_{18} \subset b_5 \cap b_8$, $a_{12} \subset b_8 \cap b_9$, $a_8 \subset b_9 \cap b_6$, $a_{14} \subset b_6 \cap b_3$, $a_1 \subset b_3 \cap b_2$, $a_9 \subset b_2 \cap b_7$, and $a_{11} \subset b_7 \cap b_{10}$. Hence $Z_{n+1} = \bigcup_{i=1}^{20} a_i$ is a continuum, and is the union of twenty continua satisfying (b).

Note that $\pi_1(a_1 \cup a_{13} \cup a_{17} \cup a_{10}) = \pi_2(a_1 \cup a_{13} \cup a_{17} \cup a_{10}) = X_1$, so (a) is also satisfied by $Z_{n=1}$. Furthermore, (e) is satisfied since

There is a z_1' in OC with (B,z_1') in $a_{14} = \langle OB,OC\rangle'$ and (z_1',B) in $a_{13} = \langle OC,OB\rangle'$.

There is a z_2' in OA with (B,z_2') in $a_1 = \langle OB,OA\rangle'$ and (z_2',B) in $a_2 = \langle OA,OB\rangle'$

There is a z_3' in OD in (B,z_3') in $a_{16} = \langle OB,OD\rangle'$ and (z_3',B) in $a_{15} = \langle OD,OB\rangle'$

As $a_i^{-1} = a_{i+1}$ for odd i, $1 \leq i \leq 19$, (c) is satisfied by Z_{n+1}.

Finally, we must show fxf $(Z_{n+1}) = Z_n$. Since fxf $(a_i) \subset Z_n$ for $1 \leq i \leq 20$, we need only show $Z_n \subset$ fxf (Z_{n+1}). We observe that fxf $(a_3 \cup a_{13} \cup a_{15} \cup a_{18} \cup a_{20})$ includes

$$\langle OB,OA\rangle,\ \langle OB,O\tfrac{A}{3}\rangle,\ \langle OC,OA\rangle,\ \langle OC,O\tfrac{A}{3}\rangle,\ \langle OD,OA\rangle$$

$$\langle OD,O\tfrac{A}{3}\rangle,\ \langle OC,OB\rangle,\ \langle OD,OB\rangle,\ \langle OD,OC\rangle,\ \langle O\tfrac{A}{3},\tfrac{5A}{6}\rangle.$$

Hence fxf $(a_4 \cup a_{14} \cup a_{16} \cup a_{17} \cup a_{19})$ includes

$$\langle OA,OB\rangle,\ \langle O\tfrac{A}{3},OB\rangle,\ \langle OA,OC\rangle,\ \langle O\tfrac{A}{3},OC\rangle,\ \langle OA,OD\rangle$$

$$\langle O\tfrac{A}{3},OD\rangle,\ \langle OB,OC\rangle,\ \langle OB,OD\rangle,\ \langle OC,OD\rangle,\ \langle \tfrac{5A}{6}A,O\tfrac{A}{3}\rangle.$$

Thus $Z_n \subset fxf\ (Z_{n+1})$. Therefore, $fxf\ (Z_{n+1}) = Z_n$.

Our inductive step is thereby completed. Since (p,q) in Z_1 implies $d(p,q) \geq \frac{1}{3}$, it follows that $\sigma f^n \geq \frac{1}{3}$, for all n and the theorem is proved. QED

Figure 3.5 represents $Z_1 \subset X_1 x X_1$ with hatched and x-ed lines.

Figure 3.7 represents $Z_2 \subset X_1 x X_1$ with hatched lines going to hatched lines and x-ed lines going to x-ed lines in Figure 5 under fxf.

Two parts of Z_3 are illustrated at the bottom of Figure 3.7.

3.2.7 Theorem.

X is nonchainable.

Proof. By Theorem 3.2.6, σX satisfies the conditions of Theorem 3.2.5. Hence we may conclude that $\sigma X > 0$. Therefore, X is nonchainable. QED

That X is nonseparating is evident. That X is indecomposable is a corollary to the proof of Theorem 3.2.3 by applying Ingram and Cook's criterion of indecomposability [1967] . Another proof that X is indecomposable follows from Theorem 3.3.2 and a theorem of Sieklucki [1968] quoted in Lemma 2.2 of Brechner and Mayer [1980a]. It is evident from the construction of X that each proper subcontinuum, being chainable, is also an arc, since all subchains are relatively straight in the covers they refine.

We observe that X has two and only two endpoints, e and f, such that e is the intersection of the tower

$$\ldots \subset D_5 \subset C_4 \subset D_3 \subset C_2 \subset D_1$$

and f is the intersection of the tower

$$\ldots \subset C_5 \subset D_4 \subset C_3 \subset D_2 \subset C_1$$

3.3. Lake-of-Wada Channels.

By Lake-of-Wada channel we mean a simple canal, defined in Brechner and Mayer [1980a] which definition is due to Sieklucki [1968] (For the original Lake-of-Wada construction see Hocking and Young [1961] , pp. 143-144.)

3.3.1 Definition.

Let X be a nonseparating continuum and let D be a set homemorphic to $[0,1)$ in $E^2 - X$, where $a: D \to [0,1)$ is a given homeomorphism. Then D will be called a simple canal in X if the following three conditions are satisfied:

(1) $\bar{D} - D \subset \text{Bd } X$

(2) For each p in D, there is a "bridge" to X; that is a crosscut to X, which (crosscut) is transverse to D, and intersects D at exactly one point.

(3) If $p_i \to \infty$ (i.e., $a(p_i) \to 1$), then there is a sequence of bridges $\{Q_{p_i}\}$ such that $Q_{p_i} \cap D = \{p_i\}$ and diam $Q_{p_i} \to 0$.

If, in addition, condition (4) holds, where (4) is

(4) $\bar{D} - D = \text{Bd } X$,

we call D a simple dense canal (s-d-c) [Sieklucki, 1968].

We can show that the embedding of X given in 3.2.2 has a s-d-c or Lake-of-Wada channel in either of two ways: by directly constructing the requisite ray in E^2-X, or by defining a chain of crosscuts $\{Q_i\}_{i=1}^{\infty}$ of E^2-X that defines a prime end E of the third kind with I(E)=P(E)=X. (See Brechner [1978] for basic definition of prime ends.) For the equivalence of these methods see Brechner and Mayer [1980a], Theorem 2.9. An embedding for which there exists a s-d-c D in E^2-X such that $\bar{D} - D = \mathrm{Bd}\, X$, is termed principal in Definition 2.6 [Brechner and Mayer, 1980a]. The embedding of X here given is consequently principal.

The first four stages of the construction of a s-d-c D are illustrated in Figure 3.4. Figure 3.2 illustrates two stages of D (the "railroad tracks"). Alternately, let Q_i be a crosscut of E^2-X such that for all $i \geq 2$, the endpoints of Q_i lie in C_i and A_i, while $Q_i \subset D_{i-1}$. Then either the odd or the even subsequence of $\{Q_i\}_{i=2}^{\infty}$ defines a prime end E such that I(E)=P(E)=X. For the even subsequence, $Q_i \to e$, and for the odd subsequence, $Q_i \to f$, where e and f are the endpoints of X. We thus have the theorem below.

3.3.2 Theorem.

X can be embedded with a simple dense canal D in E^2-X such that $\bar{D} - D = X$. That is, the embedding of 3.2.2 is principal.

3.3.3 Corollary.

X is indecomposable.

Proof. Follows from 3.3.1 and Lemma 2.2 of Brechner and Mayer [1980a].

If every embedding of X into the plane were such that E^2-X contained a s-d-c, then X would be a principal continuum in the sense of Definition 2.6 of Brechner and Mayer [1980a]. As a principal continuum X would be a candidate for a nonseparating plane continuum admitting a fixed point free map. Thus two questions suggest themselves

3.3.4 Question.

Is X a principal continuum?

3.3.5 Question.

Does X have the fixed point property for continuous maps?

In [1975] Lelek asks if there is an example of a nonchainable continuum with span zero. This author, in efforts to modify Ingram's example to produce a principally embedded, atriodic, nonchainable continuum considered several examples for which said author was not able to prove the span nonzero. On the other hand, the examples are principally embedded, and appear to be nonchainable, though atriodic. We show a schematic diagram for the bonding function for one such example as an inverse limit of T's in Figure 3.8. (For surjective span see Lelek [1977].)

3.3.6 Question.

Does the continuum of Figure 3.8 have a span greater than zero? Surjective span greater than zero?

3.3.7 Question.

Is the continuum of Figure 3.8 chainable?

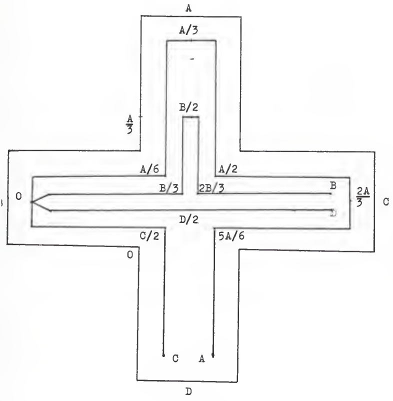


Figure 3.1

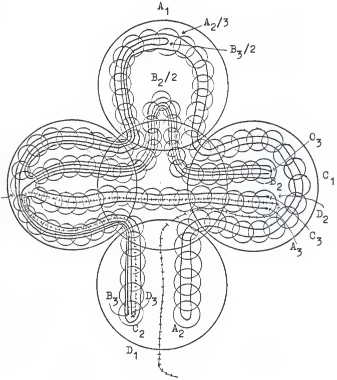


Figure 3.2

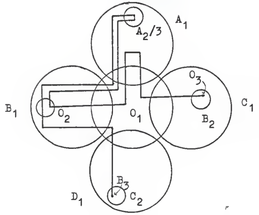


Figure 3.3

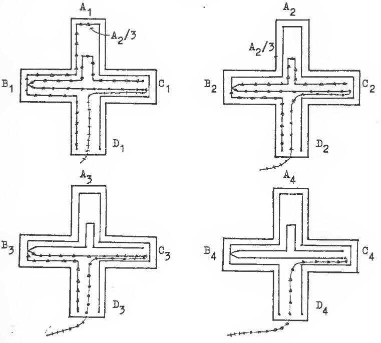


Figure 3.4

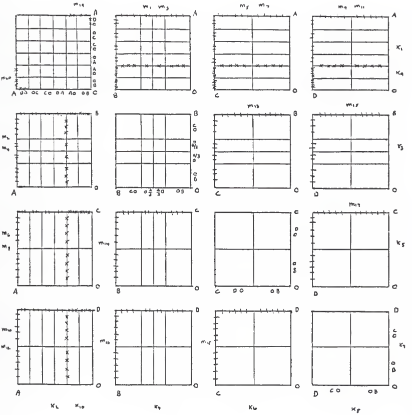
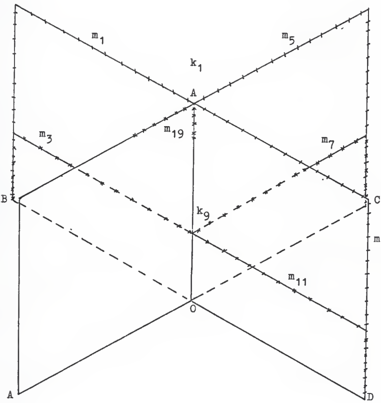

m14
m1 m3
m5 m7
m9 m11
K1
K9
m20
A
B
C
D
O
m13
m15
m2
m4
K3
m17
m6
m8
m16
K5
m10
m12
m18
m19
K2 K10
K4
K6
K8
K7

Figure 3.5



Part of $X_1 \times X_1$ with part of Z_1 marked

Figure 3.6

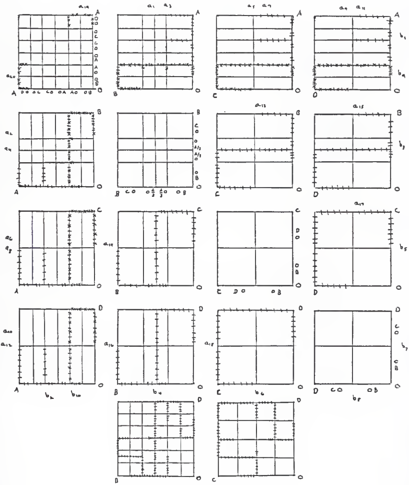

a19
A
a1 a3
a5 a7
a9 a11
b1
a20
b9
A
B
C
D
O
a13
a15
a2
a4
b3
a17
a6
a8
a14
b5
a10
a12
a16
a18
b7
b2
b10
b4
b6
b8

Figure 3.7

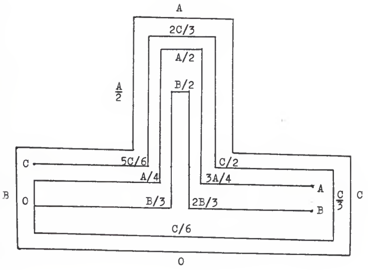


Figure 3.8

CHAPTER 4
INEQUIVALENT EMBEDDINGS AND PRIME ENDS

4.1 INTRODUCTION.

In [1980a] Brechner and Mayer show that equivalent embeddings of a nonseparating plane continuum have the same prime end structure (Theorem 2.11). Though not explicitly stated, this fact has been used previously in the literature. For instance, M. Smith [1980] and W. Lewis [1980] have independently shown that there are uncountably many inequivalent embeddings of the pseudo arc into the plane. This result was achieved by exploiting different prime end structures (directly in Lewis' case, indirectly, in terms of differing accessibility of composants by Smith) to distinguish different embeddings.

In this Chapter we show that the converse of Brechner and Mayer's theorem is false: there are inequivalent embeddings of a nonseparating continuum into the plane that have the same prime end structure, and indeed, that have the same set of accessible points. The following theorems stand in partial contrast to the methods of Smith and Lewis:

4.1.1 Theorem.

There exist uncountably many inequivalent embeddings of the sin 1/x continuum into the plane with the same prime end structure and the same set of accessible points.

4.1.2 Theorem.

There exist uncountably many inequivalent embeddings of the Knaster U-continuum (bucket handle) into the plane with the same prime end structure. Moreover, the set of accessible points in each of these embeddings is exactly the composant of the U-continuum that contains the endpoint of the U-continuum.

In Section 4.2, we show how to construct uncountably many embeddings of the continuum formed by the sin 1/x curve plus its limit segment. Thereafter, we show these embeddings have the same prime end structure, but that any two are inequivalent, thus proving Theorem 4.1.1.

In Section 4.3, we proceed similarly to prove Theorem 4.1.2, showing the Knaster U-continuum also has uncountably many inequivalent embeddings with the same prime end structure and the same set of accessible points. We first prove the theorem for two specific, easily visualizable, embeddings of the U-continuum.

In Section 4.4, we indicate how Theorem 4.1.2 can be extended to each of the uncountable class of U-type Knaster continua identified by W.T. Watkins [1980] .

4.2 The Sin 1/x Continuum.

The standard embedding of the sin 1/x continuum (Figure 4.1) consists of a ray R, the graph of (0,1] under the function y=sin 1/x in the xy-plane, plus the limit segment [p,q] , the interval [-1,1] on the y-axis. The ray R consists of a number of loops, where a loop is a segment of R with exactly one peak and one trough in the standard embedding. For simplicity, we will fix the set of loops as the segments of R between alternate successive points of zero amplitude in order from 1 to 0 on the x-axis. The endpoints of loops limit on (0,0) in the standard embedding, so on the "midpoint" (or, more precisely, some interior point) of [p,q] in any other embedding. Endpoint p of limit segment [p,q] is the limit point of points on R selected from successive troughs, endpoint q is the limit point of points selected from successive peaks. We suppose R to be coordinatized by the function $g: [0,\infty) \to R$ so that the endpoint of R is g(0) and for $x \varepsilon (0,1]$, odd positive integers correspond to $\{(x,1): \sin 1/x = 1\}$, even positive integers correspond to $\{(x,-1): \sin 1/x = -1\}$, and fractions with denominator 2 and odd numerator greater than 1 correspond to $\{(x,0): x=1/(\pi n)$, for positive integers n$\}$. This embedding, denoted K, and its corrdinatization by g are illustrated in Figure 4.1.

An embedding $e: X \to E^2$ of a continuum X is a homeomorphism into the plane; however, we shall somewhat loosely suppress reference to an embedding function and refer to the image in the plane of a continuum X as the embedding of X.

An embedding of the sin 1/x continuum can be described in terms of how the limit segment and each loop

$$L_n = [(2n-1)/2,\ n,\ (2n+1)/2,\ n+1,\ (2n+3)/2\,]$$

for odd positive integers n, is embedded in the plane. We will continue to designate the image of R, [p,q] , L_n, or a point x in the sin 1/x continuum as R, [p,q] , L_n, or x, suppressing reference to any particular embedding function. A schema will be a set of directions (necessarily infinite, though countable) for embedding (and, ambiguously, the embedding of) the loops of ray R with respect to limit segment [p,q]. A subschema will be directions for embedding some finite number of loops. Subschemata will be linked in sequence to form a schema.

We will reference the various subschemata rather informally. An example will indicate our procedure. We assume the limit segment [p,q] is fixed, and describe the embedding of the loops with reference to [p,q] . The embedding M_0 in Figure 4.2 is the simplest of the uncountably many we will show to exist. For this embedding we require only one subschema S_0: from (2n-1)/2 bend around q at n, then toward, but not around p at n+1, and extend to (2n+3)/2. The schema P_0 for embedding M_0 is then the infinite sequence of S_0's linked so that corresponding parts of successive subschemata are closer to [p,q] than their predecessors.

4.2.1 The Prime End Structures of K and M_0.

Before proceeding to construct our uncountably many embeddings of the sin 1/x continuum, we illustrate some of the concepts involved in prime end theory by applying them to K and M_0. Definitions and further references may be found in Brechner [1978] .

Prime ends are a way of looking at and classifying the approaches to the boundary of a simply connected domain with nondegenerate boundary. The complement in S^2 of a nonseparating nondegenerate plane continuum X, denoted S^2-X, is a simply connected domain. While E^2-X is not simply connected, as $E^2\cup\{\infty\}$, the one-point compactification of E^2, is S^2, we can refer to the prime end structure of E^2-X by associating it with the prime end structure of S^2-X, where the embedding at X misses the point at infinity.

A prime end of E^2-X is defined by a chain of crosscuts converging to a point of X, where a crosscut is an open arc in E^2-X whose endpoints lie in X. If Q is a crosscut of E^2-X, then $Q\cup X$ separates E^2. A sequence of crosscuts $\{Q_i\}_{i=1}^{\infty}$ is a chain provided that Q_i converges to a point, that no two crosscuts have a common endpoint, and that Q_i separates Q_{i-1} and Q_{i+1}. So, for example, the prime ends E and F of E^2-K in Figure 4.1 are defined by chains of crosscuts $\{Q_i\}_{i-1}^{\infty}$ and $\{Q_i'\}_{i=1}^{\infty}$, respectively, while in Figure 4.2, $\{T_i\}_{i=1}^{\infty}$ defines prime end H of E^2-M_0.

The impression of a prime end E, denoted I(E), is the intersection of the closures of the domains cut off by the crosscuts in a chain defining E. For example, in Figure 4.1 it can be noted that $I(E)=[p,q]=I(F)$, while in Figure 4.2, $I(H)=[p,q]$.

The set of principal points of a prime end E, denoted P(E), is the collection of all points in X to which some chain of crosscuts defining E converges. For example, in Figure 4.1, $P(E)=\{p\}$, and $P(F)=\{q\}$, while in Figure 4.2, $P(H)=\{p\}$.

A prime end E is of the first kind if $I(E)=P(E)$, both degenerate, of the second kind if $I(E)\neq P(E)$, only P(E) degenerate, of the third kind if $I(E)=P(E)$, both nondegenerate, and of the fourth kind if $I(E)\neq P(E)$, both nondegenerate. It can be shown that $P(E)\subset I(E)$ in any case, and that both are continua in X. Thus prime ends E, F, and H of Figures 4.1 and 4.2 are all of the second kind. Any other prime end G, of either E^2-K or E^2-M_0, will be of the first kind, or trivial. Thus we can say that the prime end structure of E^2-K consists of two prime ends of the second kind and all other prime ends trivial.

A more precise description of prime end structure is afforded by the notion of a C-map (see Brechner [1978].) A C-map ϕ is a homeomorphism of S^2-X onto Ext B, where Ext B is the complementary domain of the unit disk B in S^2 which contains the point at infinity, which (map) satisfies the conditions:

(1) if Q is a crosscut of S^2-X, then $\phi(Q)$ is a crosscut of Ext B, and

(2) the endpoints of images of crosscuts are dense in Bd B, the boundary of B.

If we require, as we may, that ϕ take the point at infinity in S^2-X to the point at infinity in Ext B, then we can regard the restriction ϕ to $S^2-\{\infty\}=E^2$ as a C-map also.

Suppose a chain of crosscuts defines a prime end E of E^2-X. Then the images of the crosscuts under a C-map ϕ will converge to a single point e in Bd B. We say e corresponds to E. In fact, there is a one-to-one correspondence between the prime ends of E^2-X and the points of Bd B. For example, in Figure 4.1, points e and f in Bd B correspond to prime ends E and F, respectively. In Figure 4.2, h corresponds to H. No homeomorphism of Bd B in Figure 4.1 to Bd B in Figure 4.2 can carry points corresponding to prime ends of a given kind onto points corresponding to prime ends of the same kind. Hence the prime end structures of K and M_0 are not identical. (See Brechner and Mayer [1980a], Definition 2.10.) Consequently, K and M_0 are inequivalently embedded. (See Brechner and Mayer [1980a], Theorem 2.11.)

In the general result which follows, each embedding of the sin 1/x continuum will have the same prime end structure as M_0. Note that the accessible points of M_0 are the ray R and point p of [p,q]. A point of a plane continuum X is accessible if it can be reached by a half open arc in the complement whose closure adds exactly that point. Such a half open arc is called an endcut. Each of our uncountably many embeddings of the sin 1/x continuum will have the same set of accessible points as M_0. The significance of this result is the contrast it provides to the usual procedure for producing inequivalent embeddings of a plane continuum: produce embeddings with different points accessible. Such a procedure is a sufficient, but not a necessary, condition for producing inequivalent embeddings of a continuum in E^2.

In order for subschemata to be constituents of a schema for embedding the sin 1/x continuum, the subschemata must be linkable into a ray that converges to limit segment [p,q]. Only certain subschemata are so linkable, and the following Lemma identifies countably many of them.

4.2.2 Lemma.

There exist a countable infinity of subschemata linkable pairwise in either order.

Proof. In the following, all bends in loops are either around q, or toward, but not around p, or away from p. We will describe the first few subschemata fully, and then supress reference to q-bends and designate p-bends as either in or out with a superscript for the number of bends. Let n be a positive odd integer. The subschemata are:

S_0: from (2n-1)/2 bend around q at n, then toward, but not around p at n+1, and extend to (2n+3)/2. (1 loop: in^1.)

S_1: from (2n-1)/2 bend around q at n, then away from p at n+1, extending to (2n+3)/2; then bend around q at n+2, away from p at n+3, to (2n+7)/2; then bend around q at n+4, toward, but not around p at n+5, to (2n+11)/2 which will lie between [p,q] and previous loops. (3 loops: out^2, in^1.)

S_2: 5 loops: out^4, in^1.

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S_k: 2k+1 loops: out^{2k}, in^1.

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Subschemata S_1 and S_2 are illustrated in Figure 4.3.

By keeping bends which are away from p inward of loops of the preceding subschema, and as each subschema ends at and odd half which follows a bend toward p and lies between [p,q] and all previous loops, the subschemata are linkable pairwise in either order. That is, S_k can be embedded, followed by S_m, or S_m followed by S_k, and on either side of [p,q], depending upon where the last subschema ended. Subschemata can thus be linked successively so as to limit on [p,q]. Figure 4.4 illustrates the first few stages in an embedding according to a schema whose sequence of subschemata begins S_2, S_0, S_1. QED

4.2.3 Definition.

Let N represent the set of nonnegative integers. Two infinite sequences A and B selected from N (with replacement) are distinct if after removing any finite (possibly null) initial subsequence from A and any finite (possibly null) initial subsequence from B, the remaining infinite sequences A' and B' are not identical. For example, {1, 3, 5, ...} and {2, 4, 6, ...} are distinct; {0, 0, 0, ...} and {1, 1, 1, ...} are distinct; {1, 2, 3, ...} and {4, 5, 6, ...} are not distinct. The definition of distinct sequences generalizes to infinite sequences selected from any set indexed by N.

4.2.4 Lemma.

There exist uncountably many distinct infinite sequences selected from N.

Proof. Let S be the set of all infinite sequences selected from N, which is uncountable. Partition S into equivalence classes under the following relation: $A \cong B$ iff removal of some finite (possibly null) initial subsequence from B leaves the remaining infinite sequences A' and B' identical. Since the members of an equivalence class of $S/\cong$ differ only by an initial finite subsequence, each equivalence class is countable. But S is uncountable; hence the set of equivalence classes $S/\cong$ is uncountable. Let A be an index set whose cardinality is the same as that of $S/\cong$. For each $a \varepsilon A$, let N_a be a representative of one of the equivalence classes. Then $\{N_a\}_{a \varepsilon A}$ is an uncountable collection of infinite sequences selected from N, each pair of which are distinct. QED

4.2.5 Lemma.

There exist uncountably many distinct schemata for embedding the sin 1/x continuum.

Proof. We use the notation of Lemma 4.2.2 and 4.2.4. Let $\{S_0, S_1, S_2, \ldots\}$ be the set of subschemata and let $\{N_a\}_{a \varepsilon A}$ be the set of distinct infinite sequences selected from N. Suppose that $N_a = \{a_1, a_2, a_3, \ldots\}$ and let $P_a = \{S_{a_1}, S_{a_2}, S_{a_3}, \ldots\}$ be a schema for embedding the sin 1/x continuum. It follows that $\{P_a\}_{a \varepsilon A}$ is uncountable, and that for all $a \neq b \varepsilon A$, P_a and P_b are distinct lists of subschemata. QED

4.2.6 Proof of Theorem 4.1.1.

Let $\{P_a\}_{a \varepsilon A}$ be the uncountable set of distinct schemata demonstrated in Lemma 4.2.5. For each $a \in A$, let M_a be the embedding of the sin 1/x continuum according to schema P_a. We claim the prime end structures of M_a and M_b for all $a \neq b \varepsilon A$

are identical. Any chain of crosscuts converging to p in M_a (M_b) defines a prime end E_a (E_b) of the second kind, for which $[p,q]$ is the impression and p is the only principal point. Any other prime end of M_a (M_b) is trivial. Moreover, the set of accessible points of M_a (M_b) is the ray R_a (R_b) and the point p of $[p,q]$. Hence M_a and M_b have the same prime end structure and the same set of accessible points.

We claim that M_a and M_b are inequivalent embeddings. By way of contradiction, suppose that M_a and M_b are equivalently embedded. Then there is a homeomorphism $h:E^2 \to E^2$ onto the plane such that $h(M_b)=M_a$. Then $h(R_b)=R_a$, and h must be order-preserving on R_b. Refer to Figure 4.5 in what follows.

In M_b let $\{p_j\}_{j=1}^{\infty}$ be those even integer points on R_b which lie on outer loops; that is, those integer points of R_b for which a sequence of crosscuts $\{Q_j\}_{j=1}^{\infty}$, each Q_j joining p_j to p, can be chosen so that diam $Q_j \to 0$ as $j \to \infty$. (These points are precisely the in bends of each subschema. Note that this sequence of crosscuts is not a chain.) Each pair p_j, p_{j+1} of outer loop points is separated on R_b by a finite set $\{p_j(1), p_j(2), \ldots, p_j(n_j)\}$ of even integer points of R_b which lie on inner loops. (These are precisely the out bends of each subschema.) Note that any crosscut joining $p_j(i)$ to p has diameter no less than that of $[p,q]$.

For each j, let $A_j = \{p_j, p_j(1), p_j(2), \ldots, p_j(n_j), p_{j+1}\}$ be the sequence of even integer points in order on R_b described in the preceding paragraph. Consider $h(A_j)$. Since diam $Q_j \to 0$, diam $h(Q_j) \to 0$, as well. Hence outer loop points p_j, p_{j+1} cannot infinitely often be carried to loop ends of R_a in the neighborhood of p which contain inner loop points. Similarly, inner loop points of R_b cannot infinitely often be carried onto loop ends of R_a that contain outer loop points. As the even integer points converge to p, and h is order-preserving, there is a positive integer n, such that for all $j \geq n$, the set A_j consisting of two outer loop points and n_j inner loop points between them must be carried into a set of loop ends of R_a in the close neighborhood of p consisting of two outer loop ends and n_j inner loop ends between them.

Each loop end of R_a in the neighborhood of p contains an even integer point of R_a. For simplicity we may assume that h carries, from p_n onward, even integer points of R_b to even integer points of R_a, one-to-one and order-preserving. If $\{p'_k\}_{k=1}^{\infty}$ is the set of even integer points of R_a on outer loops, then our preceding remarks require that there be a positive integer m such that there is a set of even integer points $B_m = \{p'_m, p'_m(1), p'_m(2), \ldots, p'_m(n_n), p'_{m+1}\}$ that corresponds one-to-one to the set A_n, including $p'_m(i)$ being an inner loop point. Thereafter, B_{m+1} corresponds one-to-one to A_{n+1}, B_{m+2} to A_{n+2}, etc.

Each subschema contains only one outer loop; that is, one _in_ bend "toward, but not around p." Hence A_n and B_m each correspond to a listing of the even integer points in a single subschema preceded by the even integer point on the outer loop of the preceding subschema. Likewise for A_{n+1} and B_{m+1}, etc. This implies that in P_a and P_b, $S_{a_n} = S_{b_m}$,

$S_{a_{n+1}} = S_{b_{m+1}}$, etc., and so P_a is identical to P_b after the removal of some finite initial subsequence from one and some finite initial subsequence from the other. This contradicts our choice of P_a and P_b distinct.

Therefore, no such homemorphism h can exist, and so M_a and M_b are inequivalently embedded, but have the same prime end structure and same set of accessble points. QED

4.2.7 Remark.

The uncountable class above does not exhaust the inequivalent embeddings of the sin 1/x continuum. All the embeddings we have considered have the prime end structure of Figure 4.2. There are also uncountably many inequivalent embeddings of the sin 1/x continuum with the prime end structure of Figure 4.1, and every point accessible. Additional embeddings with the prime end structure of Figure 4.1 are illustrated in Figure 4.6. In Figure 4.7 (a) we show an embedding with a prime end G defined by any chain of crosscuts converging to some point of $[p,q]$; for example, chain $\{P_i\}_{i=1}^{\infty}$. Note that $I(G)=P(G)=[p,q]$. Figure 4.7 (b) is another embedding with the same prime end structure. In these embeddings, only points of the ray R are accessible.

4.3 The Knaster U-Continuum.

The Knaster U-continuum U can be represented abstractly as the inverse limit system $\varprojlim \{A_i, f_i\}_{i=0}^{\infty}$ where each $A_i = [0,2^i]$ and each bonding function $f_i: A_{i+1} \to A_i$ is given by the rooftop function that takes 0 and 2^{i+1} to 0 and takes 2^i to 2^i, and is linear in between. See Figure 4.8 for a graph of f_i. This representation (used by Watkins [1980]) has an advantage over the usual representation with $A_i=[0,1]$ because of the following. The composant C of the endpoint $\langle 0,0,0\ldots\rangle$ of U is the set of points in U whose coordinates from some i onward are constant. We coordinate C by the function g: $[0,\infty)\to C$ defined by g(x)=

$$\langle x_0, x_1, \ldots, x_{i-1}, x, x, x, \ldots\rangle$$

for x in $[2^{i-1}, 2^i)$, for $i \geq 1$, and $g(x) = \langle x,x,x,\ldots\rangle$, for x in [0,1). Henceforth we identify the point g(x) in C with x in $[0,\infty)$.

An embedding of U into the plane is determined by taking the closure of an embedding of composant C. We may write C as the union of an increasing tower of arcs (in terms of our coordinatization): for example, the tower $\{A_i\}_{i=0}^{\infty}$ for $A_i=[0,2^i]$. An embedding of C into E^2 is then produced by specifying how each of the arcs in the tower is to be embedded in E^2 successively. Various embeddings of composant C will be distinguished by the tower of arcs in terms of which C is written, because, as we shall see, the ratio of the length of one arc of the tower to its predecessor will determine how the former arc is placed in the plane with respect to the latter.

We illustrate this process, and the method of proof of Theorem 4.1.2, by producing two embeddings of the U-continuum with exactly composant C accessible, and then showing that the two embeddings are inequivalent. Let $\{A_i\}_{i=0}^{\infty}$ and $\{B_j\}_{j=0}^{\infty}$ be towers of arcs, where $A_i=[0,2^i]$ and $B_j=[0,2^{2j}]$. Then arc A_{i+1} compared to arc A_i is two-to-one, while arc B_j compared to arc B_{j+1} is four-to-one.

Let C_A denote the endpoint composant of the U-continuum U_A embedded according to tower $\{A_i\}$ as follows: arc $A_0 = [0,1]$ is embedded in the plane as the interval $[0,1]$ on the horizontal axis. Arc $A_1 = [0,2]$ is embedded by first indentifying the first half $[0,1]$ with A_0 then embedding the remainder $[1,2]$ descending and below A_0 so that the point 2 in our coordinatization of C_A is on the vertical axis. We require A_1 to be everywhere within 1 of A_0.

Suppose arc A_i has been embedded in the plane. We extend this embedding to A_{i+1} by supposing A_i to be straightened out (unfolded) by some orientation-preserving homeomorphism of the plane which is the identity on $[0,1] \subset A_i$. We identify the first subarc $[0,2^i]$ of A_{i+1} with A_i. Then the second subarc $[2^i, 2^{i+1}]$ is embedded descending and below A_i so that when the straightening-out homeomorphism is reversed (refolded), A_{i+1} is everywhere within $1/2^i$ of A_i, and the point 2^{i+1} is on the vertical axis. Figure 4.9 illustrates this situation before A_i is refolded.

In Figure 4.11, the embedding C_A of the endpoint composant is shown carried out to several stages; the points $2, 2^2, 2^3, \ldots, 2^i, \ldots$, marked with a small x, are those points previously identified as lying on the vertical axis. The closure of C_A is then an embedding U_A of the U-continuum. We call this embedding the standard embedding of the U-continuum. It is equivalent to the "bucket handle" embedding.

Let C_B denote the endpoint composant of the U-continuum U_B embedded according to tower $\{B_j\}$ as follows: again, we embed $B_0=[0,1]$ on the horizontal axis as the unit interval. As $B_1=[0,2^2]$ is four-to-one compared to B_0, we think of B_1 as divided into four subarcs of length 1: $[0,1]$, $[1,2]$, $[2,3]$, and $[3,4]$. The first is identified with B_0, the remaining subarcs are embedded descending and running back and forth below B_0 so that B_1 is everywhere with 1 of B_0 and 2^2 is on the vertical axis. We also require 2 to be no further from endpoint 0 than 2^2 is, but to the right of the vertical axis. Figure 4.10, with j=0, can be taken to represent B_1.

Suppose arc B_j has been embedded and unfolded. We think of $B_j=[0,2^{2(j+1)}]$ as divided into four subarcs of length 2^{2j}, $[0,2^{2j}]$, $[2^{2j},2(2^{2j})]$, $[2,(2^{2j}),3(2^{2j})]$, and $[3(2^{2j}),2^{2(j+1)}]$. The first is identified with B_j; the remaining subarcs are embedded descending and running back and forth below B_j in such a way that when B_j is refolded. B_{j+1} is everywhere within $1/2^j$ of B_j. We also require that the points 2^{2j}, $3(2^{2j})$, and $2^{2(j+1)}$ be on the vertical axis, and that $2(2^{2j})$ be no further from 0 than $3(2^{2j})$, but to the right of the vertical axis.

Figure 4.10 illustrates the extension of the embedding to B_{j+1} following the unfolding of B_j. Figure 4.12 shows the embedding of C_B carried out to several stages; the points of the form 2^{2j}, $3(2^{2j})$ have been marked with a small x, and were those previously identified as lying on the vertical axis. In addition, points of the form

$3(2^{2j})+2(2^{2(j+1)})$ are marked with an x; no arc of C_B lies between them and the vertical axis. We have thereby identified a point on each "loop-end" of C_b that is not locally separated from endpoint 0. Of course other loop-ends of C_B are locally separated from 0. Among them are the points of the form $2(2^{2j})$, $3(2^{2j})\ \ 2^{2(j-1)}$, and $3(2^{2j})+3(2^{2(j-1)})$, which we have marked with a small o. Notice that each point marked x lies at the same vertical level as some point marked o in Figure 4.12, and that points marked x and o alternate in order C_B.

In Figure 4.11 we have also marked with o certain points of C_A, those of the form 2^i+2^{i-1}. In C_A as well each point marked x lies at the same vertical level as some point marked o, and such points alternate in order on C_A. These collections of points will figure significantly in our proof that U_A, the closure of C_A, and U_B, the closure of C_B, are inequivalently embedded into the plane.

4.3.1 Lemma.

Embeddings U_B and U_A have the same prime end structure. Moreoever, the set of accessible points of U_B (U_A) is exactly C_B (C_A).

Proof. We claim that any point in C_B (C_A) can be joined to endpoint 0 by a crosscut of $E^2-U_B(E^2-U_A)$ that is homotopic leaving endpoints fixed to a crosscut that lies above U_B (U_A) when the unfolding homeomorphism appropriate to some B_j (A_i) has been applied. That this is the case can be seen by examining the unmarked crosscuts to the continua in Figure 4.13. Points not on or near the vertical axis are not excepted. For instance, point c_1 in U_B in Figure 4.13 can be reach by a crosscut following the unmarked line from O almost to a_2, then following the hatched line from near a_2 to c_1. Each such point is clearly in C_B (C_A), and is accessible. Any arc to a point of U_B (U_A) from the complement would first have to intersect C_B (C_A). Hence C_B (C_A) constitutes the set of accessible points of U_B (U_A).

Now let $\phi: E^2-U_B \twoheadrightarrow \text{Ext } B$ and $\psi: E^2-U_A \twoheadrightarrow \text{Ext } B$ be C-maps, such as illustrated in Figure 4.13. Recall that each point of Bd B corresponds to a prime end of $E-U_B$ (E^2-U_A). In E^2-U_B (E^2-U_A) any prime end E defined by a chain of crosscuts converging to a point other than O will have as its impression a single point of C_B(C_A), and will thus be trivial. On the other hand, the prime end F defined by any chain of crosscuts of E^2-U_B (E^2-U_A) converging to O will have $I(F)=U_B$ ($I(F)=U_A$), and so will be of the second kind.

Let us suppose that the order of correspondence of prime ends to points on Bd B is clockwise, as shown in Figure 4.13. If g: Bd B $\twoheadrightarrow$ Bd B is any order-preserving homeomorphism such that $g(O')=O'$, then g will carry points corresponding to prime ends of a given kind onto points corresponding to prime ends of the same kind. Thus the prime end structures of U_B and U_A are identical ([Brechner and Mayer, 1980a], Definition 2.10). QED

4.3.2 Remark.

The homeomorphism g in Figure 4.13 can be extended to the closure of Ext B. It does not follow that g can be lifted through ϕ and ψ to a homeomorphism of the closure of E^2-U_B onto E^2-U_A. In fact, we shall show that there is no homeomorphism of E^2 onto E^2 that carries U_B to U_A.

4.3.3 Lemma.

If $h:E^2 \twoheadrightarrow E^2$ is a homeomorphism onto E^2 such that $h(U_B)=U_A$, then the following conditions hold:

(1) $h(C_B)=C_A$,

(2) $h(O)=O$,

(3) h is order-preserving on C_B to C_A,

(4) h induces a homeomorphism $g:\overline{\text{Ext } B} \twoheadrightarrow \overline{\text{Ext } B}$ such that if ϕ and ψ are the C-maps of Lemma 4.3.1, then

(a) $g(\phi(O)=\psi(O)$,

(b) g is order-preserving on Bd B to Bd B,

(c) if x' in Bd B corresponds to x in C_B under ϕ,

then $g(x')=y'$ in Bd B, where y' corresponds to $h(x)=y$ in C_A under ψ.

Proof. The first three conditions are obvious. As for condition (4), for x in Ext B we may define $g(x)=\psi h \phi^{-1}(x)$. For $x \in$ Bd B, we use the correspondence that ϕ and ψ set up between prime ends of U_B and U_A and points of Bd B. In this case, since every prime end of E^2-U_B and E^2-U_A has as its only principal point an accessible point of $U_B(U_A)$, then we simply define g to send x' Bd B, which corresponds to x in C_B, to that $y' \in$ Bd B which corresponds to that y in C_A for which $y=h(x)$. (See Theorem 2.3.3 of Brechner [1978] for reference.) The subconditions (a) through (c) then follow immediately. We may now regard Figure 4.13 as diagramming the functions referred to in Lemma 4.3.3. For points a_1,

a_2, b_3, etc., on C_B, each of which is the trivial impression of a prime end, the corresponding points on Bd B under ϕ are a_1', a_2', b_3', etc. Similarly, points of C_A correspond to points of Bd B under ψ. QED

4.3.4 Theorem.

Embeddings U_B and U_A are inequivalent.

Proof. Suppose by way of contradiction that the embeddings are equivalent. Then there is a homeomorphism $h:E^2 \twoheadrightarrow E^2$ such that $h(U_B)=U_A$.

In C_B we identify a sequence S_B of points in our coordinitization that lie on (or near) the vertical axis. Those are the points we have previously marked with a small x in Section 4.3, and are of the form:

$$\{4,\ 12,\ 14,\ 16,\ \ldots\ ,\ 2^{2j},\ 3(2^{2j}),\ 3(2^{2j}) + 2(2^{2(j-1)}),\ 2^{2(j+1)},\ \ldots\}$$

We divide the points of S_B into two types: those expressed without an additive term we call type a or a-points; those expressed in an additive form we call type b or b-points. With this convention. we can more simply write sequence S_B as: (See Figures 4.12 and 4.13.)

$$\{a_1,\ a_2,\ b_3,\ a_4,\ a_5,\ b_6,\ a_7,\ a_8,\ b_9,\ \ldots\}$$

Sequence S_B has the property that we can join each point of S_B to O by a crosscut Q_K^B of $E^2 - U_B$ in such a way that as $k \to \infty$ diam $Q_k^B \to 0$. For example, consider the crosscut from O to b_6 marked with x's in Figure 4.12.

Similarly, in C_A we identify a sequence S_A of points on the vertical axis which we have previously marked with x, and which are of the form:

$\{2, 4, 8, \ldots, 2^i, \ldots\}$

None of these points are in additive form, so we call them all a-points, and write S_A as:

$\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$

Sequence S_A has the property that we can join each point of S_A to O by a crosscut Q_m^A of $E^2 - U_A$ in such a way that diam $Q_m^A \to 0$ as $m \to \infty$.

Note that S_B (S_A) is in order on C_B (C_A) and is in order in terms of the vertical level approaching O from below. The corresponding sequences are in clockwise order on Bd B in Figure 4.13. In U_B (U_A), $S_B(S_A)$ converges to endpoint O.

We now address ourselves to $h(S_B)$, which lies in C_A by Lemma 4.3.3. Because h is order-preserving C_B, the images of the points in S_B will be in order on C_A. Because of the short crosscuts, $h(a_k)$ will have to lie in an arc component of the intersection of a small circular disk D centered at O and U_A which can be reached by a crosscut of $E^2 - U_A$ that lies entirely in D. This places $h(a_k)$, from some k onward, close to, and on the same small subarc of C_A as, α_m in S_A, from some m onward. The situation is symmetrical if we go from C_A to C_B under h^{-1}.

Consequently, we may assume, for simplicity, that h carries S_B from the k'-th element onwards onto S_A from the m'-th element onwards, one-to-one and order-preserving. We lose no generality by supposing that k'=1. Thus $h(a_1) = \alpha_{m'}$, and for all positive integers r, $h(a_r) = \alpha_{m'+r}$ (or if the r-th element should be a b-point, $h(b_r) = \alpha_{m'+r}$.)

Refer to Figures 4.11, 4.12, 4.13 and suppose for illustrative purposes that $h(a_1)=\alpha_3$, then $h(a_2)=\alpha_4$. Since h is order-preserving on C_B, for $[a_1,a_2]$ in C_B, we have $h([a_1,a_2])=[\alpha_3,\alpha_4]$. We now propose to establish the image of another point in the interval $[a_1,a_2]$ more precisely.

First note that the interval $[a_1,a_2]$ of C_B (almost) bounds a narrow subdomain of E^2-U_B surrounded, except for a small opening near the vertical axis, by $[a_1,a_2]$. Similar narrow subdomains are found surrounded by the interval of C_B between any two consecutive elements of S_B. There is such a narrow subdomain surrounded except, for a small opening, by the interval of C_A between any two consecutive element of S_A. We will call such intervals as $[a_1,a_2]$ pockets.

Not all pockets are alike; because of the way C_B is folded in its embedding, the pocket between a pair of consecutive a-points bends upward, so that its closed end is vertically closer to 0 than its open end, though both are close to 0. On the other hand, the pocket between an a-point and a b-point, taken consecutively in either order, bends downward, so that its closed end is at a lower vertical level than its open end. Compare the pockets $[a,_1,a_2]$ and $[a_5,b_6]$ in Figure 4.12.

Note that in U_A in Figure 4.11, all the pockets are alike in the following respect: the pocket between two consecutive points of S_A bends downward, so that its closed end is at a lower vertical level than its open end. In this respect, pockets in U_A are all like the pockets in U_B between an a-point, b-point pair.

There are basically five types of pockets which we will be able to specify more precisely once we have indentified a second sequence of points on each of C_B and C_A. We designate a sequence of points T_B on C_B, those previously marked with o, lying at the closed ends of the pockets described above. These points are of the form:

$$\{6,\ 13,\ 15,\ 32,\ \dots,\ 2(2^{2j}),\ 3(2^{2j})+2^{2(j-1)},$$

$$3(2^{2j})+3(2^{2(j-1)}),\ 2(2^{2(j+1)}),\ \dots\}$$

Those in additive form we call points of type d or d-points. Those not in additive form we call points of type c or c-points. Thus T_B can be written more simply, and mirroring S_B, as:

$$\{c_1,\ d_2,\ d_3,\ c_4,\ d_5,\ d_6,\ \dots\}$$

We remark that a sequence of crosscuts $\{P_k^B\}_{k=1}^{\infty}$ of E^2-U_B, each joining a point of T_B to O, must be such that as $k\to\infty$, diam $P_k^B\to$ diam U_B, and that $U_b \subset \limsup \{P_k^B\}$.

Similarly, we identify a sequence T_A of points of C_A, previously marked o, which lie at the closed ends of the pockets of C_A, and are of the form:

$$\{3,\ 6,\ 12,\ \dots,\ 2^i+2^{i-1},\ \dots\}$$

These are all of type d, and so we can write T_A as:

$\{\delta_1, \delta_2, \delta_3, \ldots\}$

In the order established on C_B (C_A), between each pair of consecutive points of S_B (S_A) at the open end of a pocket, there is a point of T_B (T_A) at the closed end of a pocket, and lying midway between the points of S_B (S_A) in our coordinatization. For example, in Figure 4.12, pocket $[a_1,a_2] = [a_1,c_1] \cup [c_1,a_2]$, where we call $[a_1,c_1]$ and $[c_1,a_2]$ the sides of the pocket. Note also that each point of S_B (S_A) lies at the same vertical level as some point of T_B (T_A). Both S_B (S_A) and T_B (T_A) converge to 0, though locally separated from each other by a Cantor set of intervals of U_B (U_A).

The five types of pockets are thus distinguished as falling into two groups:

(1) [a,c,a] pockets bend upward, so that their closed ends are at a higher vertical level than their open ends:

(2) [a,d,a], [a,d,b], [b,d,a], and [b,d,b] pockets bend downward, so that their closed ends are at a lower vertical level than their open ends.

We see [a,ca], [b,d,a], and [a,d,b] pockets in U_B; for examples, consider $[a_1,c_1,a_2]$, $[a_5,d_5,b_6]$, and $[b_6,d_6,a_7]$ in Figure 4.12. We see only [a,d,a] pockets in U_A: for example, consider $[\alpha_3,\alpha_4]$ in Figure 4.11. There are no [b,d,b] pockets in these two examples, though such pockets do occur in those embeddings where there is more than one b-point between some pairs of a-points; see Figure 4.16, and b_9, b_{10}, and b_{11}.

In carrying C_B to C_A homeomorphism h must carry pockets to pockets. We specify this more precisely in the following:

4.3.4.1 Claim. There is a positive integer k' such that if $a_{k'}$, $a_{k'+1}$ are in S_B and $[a_{k'}, a_{k'+1}]$ is the pocket in C_B between them with $c_{k'}$, the point of T_B in $[a_{k'}, a_{k'+1}]$, then for some positive integer m':

(1) $h(a_{k'})=\alpha_m{}'$ and $h(a_{k'=1})=\alpha_{m'+1}$,

(2) $h(c_{k'})=\delta_m{}'$,

where $\alpha_{m'}$, $\alpha_{m'+1}$ are in S_A, $[\alpha_{m'}, \alpha_{m'+1}]$ is the pocket in C_A between them, and $\delta_{m'}$ is the point of T_A in $[\alpha_{m'}, \alpha_{m'+1}]$. It further follows that:

(3) $h(\ a_{k'}, a_{k'+1}\ )=[\alpha_{m'},\ \alpha_{m'+1}]$,

(4) $h(\ a_{k'}, c_{k'}\ )=[\alpha_{m'}, \delta_{m'}]$,

(5) $h(\ c_{k'}, a_{k'+1}\ )=[\delta_{m'}, \alpha_{m'+1}]$.

Moreover, if for some positive integer r, $a_{k'+r}$, $a_{k'+r+1}$ are in S_B, then conditions (1)-(5) hold for $\alpha_{m'+r}$, α_{m+r+1} in S_A.

Proof of claim. We have already established that S_B is carried one-to-one and order-preserving onto S_A, from some m'-th element onwards. So condition (1) holds. Note that condition (2) and the fact that h is order-preserving on C_B imply conditions (3)-(5).

To establish condition (2), let x_k be a point on the vertical axis in $E - U_B$ between a_k and a_{k+1}. At the closed end of pocket $[a_k, a_{k+1}]$ lies point c_k of T_B. Let $[x_k, c_k]$ be an endcut to c_k in E^2-U_B. Endcut $[x_k, c_k]$ will be an arc in the narrow subdomain bounded by the pocket from the open end to the closed end. For example, see the hatched arc $[x_1, c_1]$ in Figure 4.12.

At each point of $[x_k, c_k]$ there is a transverse crosscut in E^2-U_B to U_B, specifically to points on the sides of pocket $[a_k, a_{k+1}]$. For any given pocket, the transverse crosscuts can be kept to a small diameter, roughly the "width" of the pocket. As $k \to \infty$, the diameter of the transverse crosscuts to the sides of the pockets can be made to go to zero.

Hence, as $k \to \infty$, the image of $[x_k, c_k]$ under h must be an endcut in E^2-U_A from a point y_m near the vertical axis, close to α_m and α_{m+1}, to a point of C_A close to δ_m in T_A, and on the same component of a small circular disk centered at O intersected with U_A as δ_m. Hence for simplicity, we may assume that from some k' onward, $h(c_{k'}) = \delta_{m'}$, for some m'; furthermore, if r is a positive integer such that $c_{k'+r}$ is in T_B, then $h(c_{k'+r}) = \delta_{m'+r}$ in T_A. QED(4.4.1)

We lose no generality by supposing that k'=1. We now proceed to show that a contradiction has arisen.

Let $[0, x_k]$ be an endcut in E^2-U_B from x_k, identified in Claim 4.3.4.1, to endpoint O of U_B. Then $[0, x_k, c_k]$ is a crosscut of E^2-U_B joining O to c_k. We let $[0, x_k]$ be close to the vertical axis, and can require that diam $[0, x_k] \to 0$ as $k \to \infty$. (Recall that diam $[x_k, c_k]$ cannot go to zero.)

At the same vertical level as c_k there is a member of S_B. Since c_k is a c-point, the member of S_B must be a b-point, namely b_{k+5}. Recall that c_k will be at a higher vertical level than a_k and a_{k+1}. Let $[0,b_{k+5}]$ be a crosscut of E^2-U_B, such that as $k\to\infty$, diam $[0,b_{k+5}]\to 0$. For example, the x-ed crosscut in Figures 4.12 and 4.13 goes from 0 to b_6 at the same vertical level as c_1. If we join c_k to b_{k+5} by a short arc (not a crosscut) parallel to the horizontal axis, then diam $[0,b_{k+5},c_k]\to 0$ as $k\to\infty$. Furthermore, $[0,x_k,c_k]$ and $[0,b_{k+5},c_k]$ intersect only at their endpoints.

There are several ways to show that a contradiction has now arisen; we shall make use of the C-maps ϕ and ψ of Lemma 4.3.3 to do so. Let ϕ, ψ, and g induced by h, be as in Lemma 4.3.3. Figure 4.13 illustrates the maps involved.

Consider the images $h([0,x_k,c_k])$ and $h([0,b_{k+5},c_k])$: for some m, $h(x_k)=y$ which is close to $h(a_k)=\alpha_m$ and $h(a_{k+1})=\alpha m+1$. Since diam $[0,x_k]\to 0$, diam $[0,y_m]\to 0$. We have already established that $h([x_k,c_k])=[y_m,\delta_m]$ for δ_m between α_m and α_{m+1} at the closed end of pocket $[\alpha_m, \alpha_{m+1}]$. Since diam $[0,y_m]\to 0$, crosscut $[0,y_m,\delta_m]$ cannot go around U_A, but must be drawn more or less as shown by the hatched lines in Figures 4.11 and 4.13 for $[0,y_3,\delta_3]$, the image of $[0,x_1,c_1]$ in Figures 4.12 and 4.13. Since diam $[0,b_{k+5},c_k]\to 0$, so diam $[0,\alpha_t,\delta_m]\to 0$, where $\alpha_t=h(b_{k+5})$. Also, $[0,\alpha_t,\delta_m]$ cannot intersect $[0, y_m, \delta_m]$ except at 0 and δ_m. Hence, $[0, \alpha t]$ must be drawn more or less as shown by the x-ed lines in Figures 4.11 and 4.13.

As we have indicated before, each point of C_B (C_A) in order from 0 to ∞ corresponds to a point of Bd B under $\phi(\psi)$ in clockwise order from 0'. In Figure 4.13, hatched and x-ed corsscuts are carried to hatched and x-ed crosscuts by the functions h, g, ϕ, and ψ. Since α_t must be at the same vertical level as δ_m, our illustration in Figure 4.13 is appropriate for any index. However, as the reversed order of containment of hatched and x-ed crosscuts in Figure 4.13 illustrates, no such homeomorphism g, induced by h as we described, can exist.

In view of this contradiction, there cannot be a homeomorphism h from E^2 such that $h(U_b)=U_A$. Consequently, U_B and U_A are inequivalently embedded. QED(4.3.4)

4.3.5 Summary And An Observation.

The critical element of the above proof is that an [a,c,a] pocket cannot infinitely often be carried by a homeomorphism of the plane that carries U_B onto U_A onto an [a,d,a] pocket. The distinction that makes all the difference is that the [a,c,a] pocket bends upward, having its closed end at a higher vertical level than its open end, while the [a,d,a] pocket bends downward, having its closed end at a lower vertical level than its open end.

What should we expect if we have a homeomorphism of the plane onto itself that carries U_B onto itself? It need not be the identity on U_B, but it would have to carry [a,c,a] pockets to [a,c,a] pockets, and consequently, [a,d,b] pockets to [a,d,b] pockets and [b,d,a] pockets to [b,d,a] pockets, all inorder on C_B. Thus, in some sense, it would have to be a "stretching" or "compressing" of our coordinatization of C_B by some 2^k factor, k even.

There is a homeomorphism of U_B onto U_B defined by applying the two-to-one map to each factor space of a four-to-one inverse limit definition of U. Our remarks about pockets above imply that this map cannot be extended from U_B to the plane. On the other hand, the four-to-one map on the factor spaces can be extended to the plane.

4.3.6 Uncountably Many Embeddings of the U-Continuum.

In what follows we shall show that uncountably many embeddings of the U-continuum can be constructed, each with the same prime end structure and with every point of the endpoint composant, and only those points, accessible. Subsequently, we shall show that any two of these embeddings are inequivalent.

Let $\{m_i\}_{i=1}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$ be two distinct sequences of positive integers selected from $N = \{2, 3, 4, 5, \ldots\}$. That we have omitted 0,1 does not alter the conclusion of Lemma 4.2.4, but it does eliminate [a,d,a] pockets from our embeddings, thus slightly simplifying the proofs to follow. Since $\{m_i\}$ and $\{n_j\}$ are distinct, for any i' and j', we have $\{m_i\}_{i=i'}^{\infty} \neq \{n_j\}_{j=j'}^{\infty}$. Let $m_0 = n_0 = 0$. We form two sequences increasing in the nonnegative integers as follows:

$$\{k_i\}_{i=0}^{\infty}=\{m_0,\ m_0+m_1,\ m_0+m_1+m_2,\ \ldots,\ \sum_{r=0}^{i} m_r=k_{i-1}+m_i,\ \ldots\}$$

$$\{l_j\}_{j=0}^{\infty}=\{n_0,\ n_0+n_1,\ n_0+n_1+n_2,\ \ldots,\ \sum_{r=0}^{i} n_r=l_{j-1}+n_j,\ \ldots\}$$

Let $\{D_i\}_{i=0}^{\infty}$ and $\{C_j\}_{j=0}^{\infty}$ be increasing towers of arcs where for each i, $D_i=[0,2^{k_i}]$, and for each j, $C_j=[0,2^{k_j}]$.

We may consider $\{D_i\}$ and $\{C_j\}$ to be ways of writing the endpoint composant C of the U-continuum U. We shall then show how to embed C into E^2 according to tower $\{D_i\}$ ($\{C_j\}$), producing thereby embedding C_D (C_C) of C and, by taking the closure, embedding U_D (U_C) of U. Our procedure is essentially no different from that followed in the special cases preceding. We apply it to $\{D_i\}$. It should be clear thereafter how to apply the method to $\{C_j\}$ or to any tower determined by a sequence of positive integers and the increasins sequence defined therefrom. Figures 4.14 through 4.17 illustrate the procedures to be followed.

First $D_0=[0,1]$ is embedded as the unit interval on the horizontal axis of E^2. Compared in length to D_0, $D_1=[0,2^{k_1}]$ $=[0,2^{k_0+m_1}]$ is 2^{m_1} -to-one. We view D_1 as divided into 2^{m_1} subarcs, each of length 1. Identify the first subarc with D_0; the remaining subarcs are embedded descending and running back and forth below D_0. Figure 4.14 may be so interpreted with i=0 and $m_{i+1}=3$. We require that D_1 be everywhere within 1 of D_0, and that even multiples of $2^{k_0}=1$ up to $2^{m_1}-2$ be to the right of the vertical axis, but no further from 0 than 2^{k_1} which is to be on the vertical axis.

Line up the odd multiples of 2^{k_0} up to $2^{m_1}-1$ vertically below 2^{k_0}.

Suppose this embedding has been extended to D_i. We assume D_i has been straightened out (unfolded) by some orientation-preserving homeomorphism of E^2 keeping [0,1] in D_i fixed. Compared in length to $D_i=[0,2^{k_i}]$, $D_{i+1}=[0,2^{k_{i+1}}]=[0,2^{k_i+m_{i+1}}]$ is $2^{m_{i+1}}$-to-one. We view D_{i+1} as divided into $2^{m_{i+1}}$ subarcs, each of length 2^{k_i}. Identify the first subarc with D_i; the remaining subarcs are embedded descending and running back and forth below D_i, as shown in Figure 4.14 for $m_{i+1}=3$ and $m_i=2$. The embedding is carried out in such a way that D_{i+1} is everywhere within $1/2^i$ of D_i, when the straightening-out homeomorphism is reversed (refolded.) Furthermore, odd multiples of 2^{k_i} up to $2^{m_{i+1}}-1$ (the bends in D_{i+1} on the right in Figure 4.14, marked with a small x) are to be on the vertical axis when D_i is refolded, as is $2^{k_{i+1}}$. The even multiples of 2^{k_i} up to $2^{m_{i+1}}-2$ (the bends in D_{i+1} on the left in Figure 4.14, marked with a small o) are to be to the right of vertical axis, when D_i is refolded, but no further from 0 than $2^{k_{i+1}}$ is.

In marking certain points with x and o, our aim is to identify a set of points that end up near or on the vertical axis and converge to 0. Those marked x are not locally separated from the vertical axis; those marked o are locally separated from the vertical axis as we embed further arcs of the tower. So far we have marked even and odd multiples of

2^{k_i}. We also mark an additional set of points with x and o that lie on the last and (before refolding) lowest subarc of D_{i+1}. These are of the form $(2^{m_{i+1}}-1)(2^{k_i})$ plus an even or an odd multiple of $2^{k_{i-1}}$, and are marked oppositely to those points of D_i that lie directly above them. This first set of points marked x and o will be our a-points and c-points of this embedding, and the second set of points marked x and o will be the b-points and d-points.

Points marked x, when taken in a consecutive pair in order on C_D will signify the open end of a pocket, while points marked o will individually signify the closed end of a pocket. To see that this is the case, we consider the refolding of D_i, after D_{i+1} has been embedded relative to D_i, as if it proceeded one stage back at a time: first refolding to the configuration of D_{i-1}, then to the configuration of D_{i-2}, and so forth. It will be seen that points marked x and o on D_{i+1} will be, within two stages of refolding, near the vertical axis, and that further refolding will change neither their location near the vertical axis, nor their vertical orientation with respect to O, that is, their vertical order.

In Figure 4.15 we indicate how D_{i+1} will move when D_i is refolded one stage back to the configuration of D_{i-1}. The section marked A will remain on top; section B will be below A and "upside down"; section C will be below B and "rightside up"; section D will be below C and "upside down". The actual number of sections depends upon m_1, which we have assumed is 2 for illustrative purposes. At the right of section D are those points of the first $2^{m_{i+1}}-1$ subarcs marked x; they will end up, after one stage of refolding, on the vertical axis in numerical order going up toward O. At the left of section A are those points of the first $2^{m_{i+1}}-1$ subarcs of D_{i+1} marked o; they will remain near the vertical axis, upon refolding one stage back, and in order going down from O.

Those points of the last subarc of D_{i+1} marked x will also end up near the vertical axis, upon refolding one stage back to the configuration of D_{i-1}. However, the points of the last subarc of D_{i+1} marked o, when in the configuration of the D_{i-1} diagram, will be in the rightmost section. We may regard Figure 4.16 as representing the result of refolding D_i to the configuration of D_{i-1}. In refolding D_{i-1} one stage back to the configuration of D_{i-2}, we first section D_{i-1} into $2^{m_{i-1}}$ sections. Note that all points on D_{i+1} marked x, and all points on D_{i+1} marked o, except for those on the last subarc, are in the first section of the D_{i-1} diagram. (Figure 4.16 does not show the sectioning.) The points of the last subarc of D_{i+1} marked o are in the last section of the D_{i+1} diagram; for example, d_4 and d_6 in Figure 4.16.

Upon refolding the D_{i-1} diagram one stage back to the configuration of D_{i-2}, we see that those points that have arrived in the first section will remain i the first section of the next, D_{i-3}, diagram. Once a point marked x or o has gotten near the vertical axis, and in a certain vertical order, it does not change that orientation. This is because the first section of any diagram remains on top and in a fixed vertical order. Also, upon refolding the D_{i-1} diagram one stage back, we see that, finally, the points marked o on the last subarc of D_{i+1} are moved into the first section of the D_{i-2} diagram. They move, along with the points of the first $2^{m_i}-1$ subarcs of D_i marked x, to a location near the vertical axis, but locally separated from it by, among other arcs, an arc of D_i containing a point marked x.

This process generates pockets in the composant C_D. In further refolding, the critical features of these pockets, to which we draw attention below, will not change.

Consider a consecutive pair of points marked x on the first $2^{m_{i+1}}-1$ subarcs of D_{i+1}. There is a point marked o midway between them. After one stage of refolding, this pocket can be seen to bend upward, leaving its closed end, marked o, at a higher vertical level than its open end, between the two points marked x on the vertical axis. Since both closed and open ends are now in the first section of the D_{i-1} diagram, further refolding does not alter this orientation of closed and open ends. These will be the [a,c,a] pockets of this embedding.

Consider a pair of points marked x on the last subarc of D_{i+1}. We may take our choice from all the points marked x on this subarc, including both ends, provided we take consecutive pair. The pocket between them, after two stages of refolding, can be seen to bend downward, leaving its closed end, marked o, at a lower vertical level than its open end. Since both closed and open ends are now in the first section of the D_{i-2} diagram, they will not change their vertical orientation upon refolding further. These will be the [a,d,b], [b,d,b], and [b,d,a] pockets of this embedding. Because D_i is at least four-to-one compared to D_{i-1}, there will be no [a,d,a] pockets in any of these embeddings. If D_i is just four-to-one compared to D_{i-1}, then there is no [b,d,b] pocket contributed by the last subarc of D_{i+1}.

In summary, the pockets formed by the first $2^{m_{i+1}}-1$ subarcs of D_{i+1} bend upward with closed ends at a higher vertical level than open ends, and are all of type [a,c,a]. Pockets formed by the last subarc of D_{i+1} bend downward with closed ends at a lower vertical level than open ends, and are of types [a,d,b], [b,d,b], and [b,d,a].

If we assume that $m_1=m_3=2$ and $m_2=3$, then Figure 4.16 illustrates C_D. Those points marked x comprise the a-and b-points of C_D and together constitute sequence S_D, in our coordinatization, is given by repeated subsequences of the form:

$$\begin{aligned}\{\ldots,\ &2^{k_i},\ 3(2^{k_i}),\ \ldots,\ (2^{m_{i+1}}-1)(2^{k_i}),\\ &(2^{m_{i+1}}-1)(2^{k_i})+2(2^{k_{i-1}}),\ \ldots,\\ &(2^{m_{i+1}}-1)(2^{k_i})+(2^{m_i}-2)(2^{k_{i-1}}),\ 2^{k_{i+1}},\ \ldots\}\end{aligned}$$

Those in additive form are b-points and those not in additive form are a-points. Note that a sequence of a-points is followed by a sequence of b-points, and then the first a-point of the next subsequence of S_D. In our example, Figure 4.16, S_D can be written more simply as:

$$\{a_1, a_2, a_3, a_4, b_5, a_6, a_7, b_8, b_9, b_{10}, a_{11}, \ldots\}$$

Note that there are three b-points between a_7 and a_{11} because there were four a-points (and so three c-points) in the preceding subsequence. The sequence S_D can be written in the simpler form to any desired length on the basis of the m_i's, inductively.

If we assume that $n_1=n_2=n_3=2$, then Figure 4.17 illustrates embedding C_C and sequence S_C can be written:

$\{\alpha_1,\alpha_2,\beta_3,\alpha_4,\alpha_5,\beta_6,\alpha_7, \ldots$

with α_i indicating an a-point and β_i indicating a b-point.

Between each pair of consecutive points of S_D (S_C) lies a c-point or d-point that we have previously marked o. These constitute the sequence T_D (T_C). Sequence T_D is given by repeated subsequences of the form:

$$\{\ldots, 2(2^{k_i}), \ldots, (2^{m_{i+1}}-2)(2^{k_i}),$$

$$(2^{m_{i+1}}-1)(2^{k_i})+2^{k_{i-1}}, \ldots,$$

$$(2^{m_{i+1}}-1)(2^{k_i})+(2^{m_i}-1)(2^{k_{i-1}}), \ldots\}$$

Those in additive form are d-points and those not in additive form are c-points. In Figure 4.16, T_D can be written more simply as:

$\{c_1, c_2, c_3, d_4, d_5, c_6, d_7, d_8, d_9, d_{10}, \ldots\}$

Sequence T_C for embedding C_C will have c-points indicated by γ_i and d-points indicated by δ_i. With the assumptions of Figure 4.17, T_C can be written:

$\gamma_1, \delta_2, \delta_3, \gamma_4, \delta_5, \delta_6, \ldots$

Each sequence of positive integers results in an embedding of composant C of the U-continuum. That there are uncountably many such embeddings follows from the uncountability of the collection of distinct sequences. It remains to be shown that any two of these embeddings are inequivalent.

4.3.7 Proof of Theorem 4.1.2.

Suppose that U_D and U_C are two of the uncountably many embeddings that can be constructed in accordance with Section 4.3.6. We follow the notation of that section. The proof of Theorem 4.1.2 then divides into establishing two lemmas.

4.3.7.1 Lemma. Embeddings U_C and U_D have the same prime end structure and the same set of accessible points, namely the composant C_D (C_C) of the endpoint O.

Proof. As in the proof of Lemma 4.3.1, we define C-maps $\phi : E^2 - U_D \twoheadrightarrow \text{Ext } B$ and $\psi : E^2 - U_C \twoheadrightarrow \text{Ext } B$. It may then be shown that all, and only, points of C_D (C_C) can be joined to O by a crosscut in $E^2 - U_D$ $(E^2 - U_C)$ that lies above U_D (U_C) when D_i (C_j) has been unfolded, for some i (j). That to every point of Bd B which represents a prime end of U_D, there corresponds a point of Bd B which represents a prime end of U_C of the same kind, and that this correspondence is a homeomorphism of Bd B onto Bd B, follows from considerations similar to those of Lemma 4.3.1. QED(4.3.7.1)

4.3.7.2 Lemma. Embeddings U_D and U_C are inequivalent.

Proof. Suppose $h:E^2 \twoheadrightarrow E^2$ is a homeomorphism onto E^2 such that $h(U_D)=U_C$. A version of Lemma 4.3.3 then holds: $h(C_d)=C_C$, $h(0)=0$, h is order-preserving on C_D, and h induces, through C-maps ϕ and ψ , a homeomorphism $g: \overline{\text{Ext } B} \twoheadrightarrow \overline{\text{Ext } B}$. If S_D and S_C are the sequence of a- and b-points of C_D and C_C, as detailed in Section 4.3.6, then we may assume that h carries S_D from $a_1=2^1$ onward onto S_C from some element of S_C onward, and that this correspondence is one-to-one and order-preserving on both C_D and the vertical axis approaching 0 from below.

As we noted in Section 4.3.6, the pocket formed by the interval $[a_k, a_{k+1}]$ between two points of S_D is an $[a,c,a]$ pocket, bends upwards, so that its closed end containing $c_k \in T_D$ is at a higher vertical level than its open end. On the other hand, the pocket formed by $[a_k, d_k, b_{k+1}]$, $[b_k, d_k, b_{k+1}]$, or $[b_k, d_k, a_{k+1}]$ bends downward, so that its closed end is lower than its open end. As $k \to \infty$, $[a_k, c_k, a_{k+1}]$ pockets cannot infinitely often be carried onto downward bending pockets in C_C, those of the form $[\alpha_m, \delta_m, \beta_{m+1}]$, or $[\beta_m, \delta_m, \beta_{m+1}]$, but after some k, must always go onto a pocket of the form $[\beta_m, \delta_m, \alpha_{m+1}]$; that is, an $[a,c,a]$.

We get the same kind of contradiction as in the proof of Lemma 4.3.3. Examples of crosscuts which can be used to show this are illustrated in Figures 4.16 and 4.17, where it is supposed that the hatched crosscut goes to the hatched crosscut and the x-ed crosscut goes to the x-ed crosscut, under h. Hence we may assume that starting with k=1 and m=m', consecutive a-points of C_D must be carried onto consecutive a-points of C_C.

It then follows that n consecutive a-points of C_D must be carried onto a like number of consecutive a-points of C_C. Also, the b-points between a pair of a-points in C_D must be carried onto a like number of b-points of C_C between two a-points of C_C. Consequently, the subsequence of S_D of the form:

$$\{a_1, a_2, \dots, a_r, b_{r+1}, \dots, b_{r+s-1}, a_{r+s}\}$$

must be carried onto a subsequence of S_C of the form:

$$\{\alpha_{m'}, \dots, \alpha_{m'+r}, \beta_{m'+r+1}, \dots, \beta_{m'+r+s-1}, \alpha_{m'+r+s}\}$$

The same must hold for the next subsequence $\{a_{r+s}, \dots, a_{r+s+t})$ onto $\{\alpha_{m'+r+s}, \dots, \alpha_{m'+r+s+t}\}$, and so forth.

Now $a_1=2^{k_1}$ and $\alpha_{m'}=2^{l_{j'}}$, for some j'. Hence h must carry 2^{k_1} onto $2^{l_{j'}}$, 2^{k_2} onto $2^{l_{j'+1}}$, and so forth. The number of intervening a- and b-points must match exactly. But this implies that $m_1=n_{j'}$, $m_2=n_{j'+1}$, $m_3=n_{j'+3}$, and so forth. Hence the sequences $\{m_i\}_{i=1}^{\infty}$ and $\{n_j\}_{j=j'}^{\infty}$ are identical. This contradicts our assumption that sequences

$\{m_i\}_{i=1}^{\infty}$ and $\{n_j\}_{j=1}^{\infty}$, used to construct U_D and U_C, are distinct.

Hence, no homeomorphism h from E^2 onto E^2 exists that carries U_D to U_C. So U_D and U_C are inequivalently embedded. QED (4.3.7.2)

Putting Lemma 4.3.7.1 and 4.3.7.2 together, we have shown that there exist uncountably many embeddings of the Knaster U-continuum into the plane with the same prime end structure, and with exactly the endpoint composant the set of accessible points. QED (4.3.7)

4.3.8 Remark.

In the above proofs we did not expressly show that the closure of each of our embeddings of a tower of arcs actually resulted in a U-continuum. However, our requirement that arc A_{i+1} be within $1/2^i$ of arc A_i (and similarly for the other towers) permits us to produce a defining sequence of chain covers by open disks of the ray embedded according to tower $\{A_i\}_{i=0}^{\infty}$. We can arrange that each chain is 2^n-to-one in its predecessor, for some n, and so the intersection determines the U-continuum. Each A_i will then be the intersection of certain initial subchains of the covering chains, with the union of the initial subchains dense in the continuum.

We could have proceeded from defining sequences of chains of open disks in the plane to identifying composant C of the endpoint, and then defined our embeddings based upon different embedding patterns for the chains. For instance, U_A is produced by a defining sequence of chains where each chain is two-to-one and descending in its predecessor, while U_B is produced by a defining sequence of four-to-one descending chains.

4.4. Uncountably Many Embeddings of Uncountably Many Continua.

In [1980] Will Watkins has shown that there are uncountably many nonhomeomorphic Knaster continua, answering a question posed by J. M. Rogers, Jr. in [1970]. We have considered only the most basic of those, namely D_2, that one which is produced as the inverse limit of arcs under two-to-one bonding maps (or compositions thereof.) If we were to use instead six-to-one bonding maps, or compositions thereof, the resulting continuum, D_6, would not be homeomorphic to D_2, though it would be like D_2 in having a single endpoint.

4.4.1 Natural Embeddings of Knaster Continua.

Each of our embeddings of D_2 can be thought of as a "natural" embedding of the inverse limit system

$$D_2^A = \varprojlim \{A_i, f_{n_i}\}_{i=0}^{\infty}$$

where $f_{n_i}: A_{i+1} \to A_i$ is a 2^{n_i}-to-one rooftop function, $A_i = [0, 2^{k_i}]$ for $k_i = \sum_{j=0}^{i} n_j$, and $A = \{n_i\}_{i=1}^{\infty}$ is a sequence of positive integers with $n_0 = 0$. The naturalness of the embedding lies in the fact that the embedding of arc A_{i+1} relative to arc A_i is just the graph of the bonding function for the i-th stage.

In Figure 4.18 we show the natural embedding of D_6 where $D_6 = \varprojlim \{B_i, g_i\}_{i=0}^{\infty}$, $g_i : B_{i+1} \to B_i$ is a six-to-one rooftop function, and $B_i = [0, 6^i]$. Other embeddings of D_6 can be produced as the natural embedding of

$$D_6^B = \varprojlim \{B_i, g_{n_i}\}_{i=0}^{\infty}$$

where g_{n_i} is the composition of n_i six-to-one rooftop functions and B, B_i, n_i, and k_i are all analogous to the case for D_2^A. Since the use of different intervals for each A_i (B_i) is a matter of convenience in coordinatization of the endpoint composant, we could just as well let $A_i = [0,1] = B_i$.

If A and B are distinct sequences of positive integers, then D_6^A and D_6^B are homeomorphic continua, since the bonding maps are in both cases compositions of six-to-one maps. The natural embeddings of D_6^A and D_6^B can be shown, using techniques similar to those used in the proofs of Theorems 4.3.4 and 4.1.2, to be inequivalent, but to have the same prime end structure, and the same set of accessible points, namely the entire composant of the endpoint.

Watkins [1980] has shown that to each infinite subset of the primes there corresponds a Knaster continuum. Let $D_A = \varprojlim \{A_i, g_{a_i}\}_{i=0}^{\infty}$, where $A_i = [0,1]$, $A = \{a_i\}_{i=0}^{\infty}$ is a listing of primes in which each member of A', an infinite subset of the primes, appears infinitely often, and $g_{a_i} : A_{i+1} \to A_i$ is an a_i-to-one rooftop function. For each $A' \neq B'$, D_A and D_B are nonhomeomorphic. If the prime 2 should appear in A', then D_A is a Knaster continuum with one endpoint, a U-type continuum. If 2 should not appear in A', then D_A is a Knaster continuum with two endpoints, an S-type continuum. Watkins shows this entire collection of continua is uncountable, as each different A' produces a different continuum.

If we restrict our attention to those subsets of the primes not including 2, but let each $g_{a_i}:A_{i+1} \to A_i$ be a $2a_i$-to-one rooftop function, then we still have uncountably many nonhomeomorphic continua, but all U-type continua.

Let $B=\{n_j\}_{j=0}^{\infty}$ be such that $n_0=0$ and the remaining n_j's are positive integers. Let $h_0=g_{a_0}$, $h_1=g_{a_{n_1}}\circ\ldots\circ g_{a_2}\circ g_{a_1}$, $h_2=g_{a_{n_1+n_2}}\circ\ldots\circ g_{a_{n_1+1}}$, etc. That is, each h_j is the composition of the next n_j unused g_{a_i} maps. Then the inverse limit

$$D_A^B = \varprojlim \{A_j, h_j\}_{j=0}^{\infty}$$

is homeomorphic to D_A. However, if B and C are distinct sequences of positive integers, then we can show that the natural embeddings of D_A^B and D_A^C are inequivalent, though they have the same prime end structure, and have the endpoint composant as the set of accessible points. Since the bonding maps in each are different compositions of the sequence of maps defining D_A, they are homeomorphic continua.

4.4.2 Questions.

We conclude with several questions concerning extensions of Theorem 4.1.2 to additional continua.

4.4.2.1 Question. Are there uncountably many inequivalent embeddings of the pseudo arc with the same prime end structure?

4.4.2.2 Question. Are there uncountably many inequivalent embeddings of every indecomposable chainable continuum (with the same prime end structure)?

4.4.2.3 Question. Are there uncountably many inequivalent embeddings of every (of some) indecomposable nonchainable continuum (with the same prime end structure)?

4.4.2.4 Question. Does Ingram's atriodic nonchainable tree-like continuum have uncountably many inequivalent embeddings (with the same prime end structure)? (See Ingram, [1972] for the construction.)

4.4.2.5 Question. Does the atriodic nonchainable treelike continuum of Mayer [1980a] and Chapter 3, have uncountably many inequivalent embeddings with the same prime end structure, specifically, all with a simple dense canal? Do all embeddings of that continuum have a simple dense canal?

We assume all the embeddings refered to above are into the plane. Answers to Questions 4.4.2.4 and 4.4.2.5 in particular may be of some relevance to the fixed point problem for nonseparating plane continua.

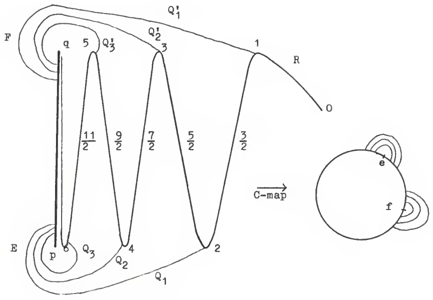

Q'_1
F
q
5
Q'_3
Q'_2
3
1
R
0
$\frac{11}{2}$
$\frac{9}{2}$
$\frac{7}{2}$
$\frac{5}{2}$
$\frac{3}{2}$
C-map
e
f
E
p
6
Q_3
4
Q_2
2
Q_1

Figure 4.1

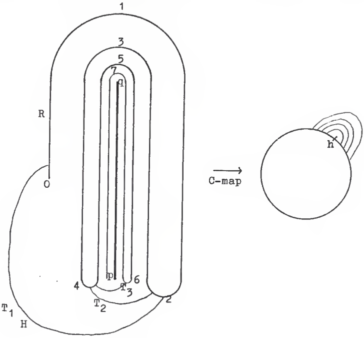


Figure 4.2

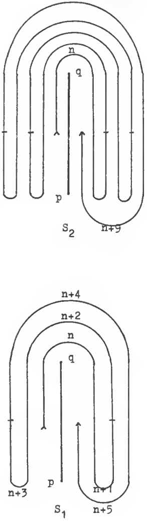
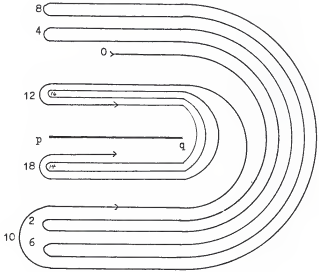


Figure 4.3



S_2, S_0, S_1

Figure 4.4

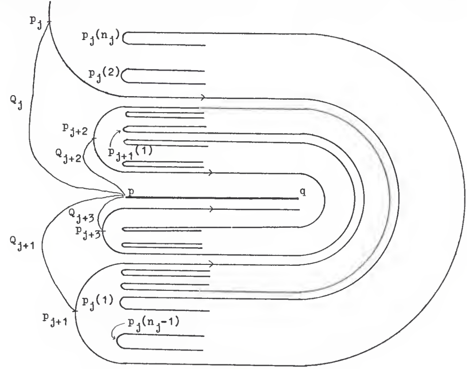
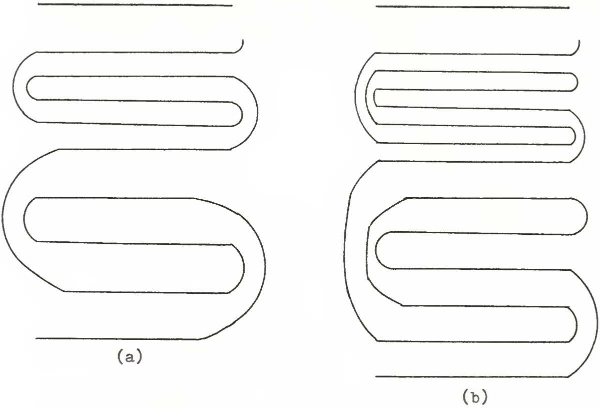


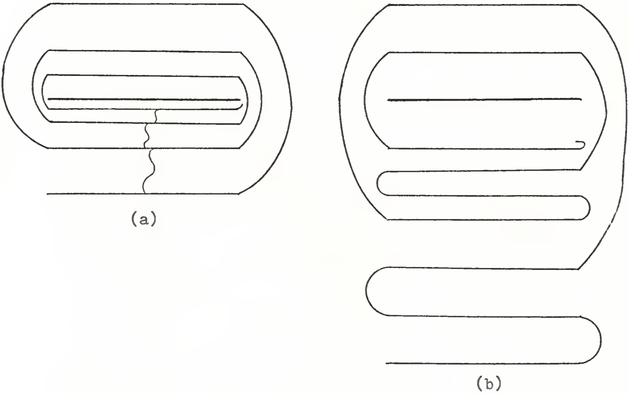
Figure 4.5



(a)

(b)

Figure 4.6



(a)

(b)

Figure 4.7

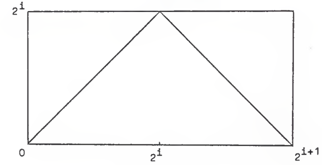


Figure 4.8

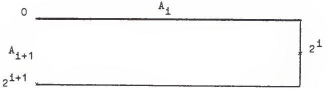


Figure 4.9

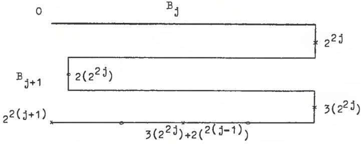
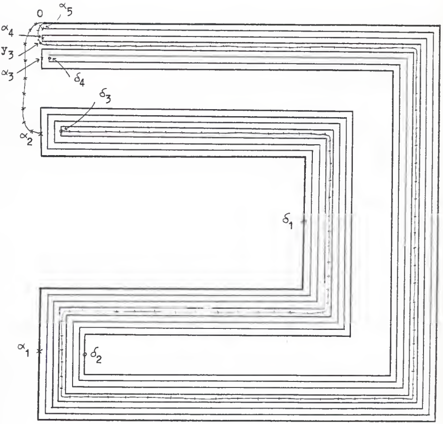


Figure 4.10



$\alpha_1 = 2^1$
$\alpha_2 = 2^2$
$\alpha_3 = 2^3$
$\alpha_4 = 2^4$
$\alpha_5 = 2^5$

$\delta_1 = 2^1 + 2^0$
$\delta_2 = 2^2 + 2^1$
$\delta_3 = 2^3 + 2^2$
$\delta_4 = 2^4 + 2^3$

Figure 4.11

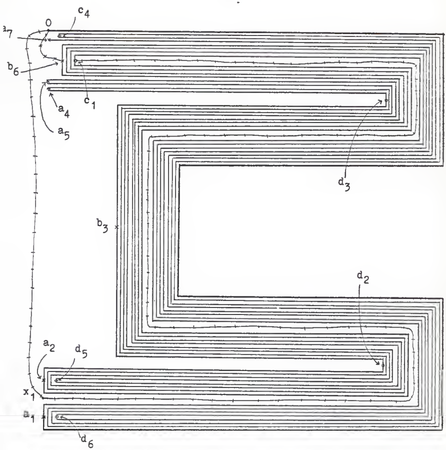


Figure 4.12

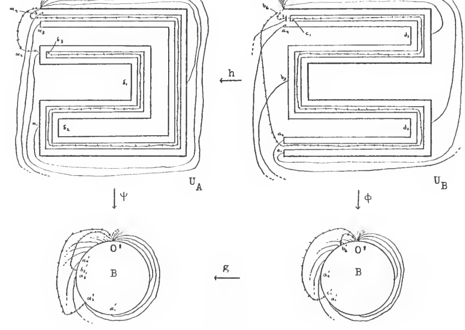


Figure 4.13

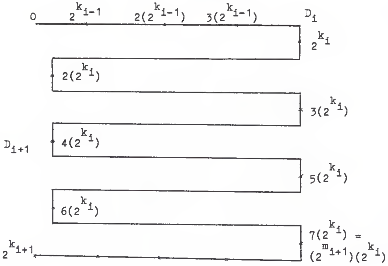


Figure 4.14

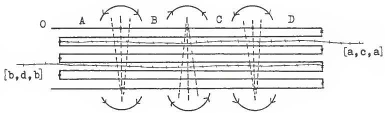


Figure 4.15

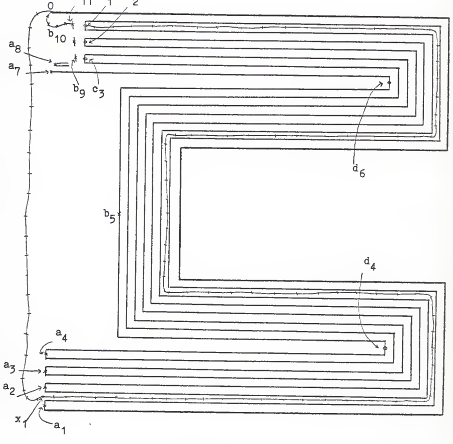
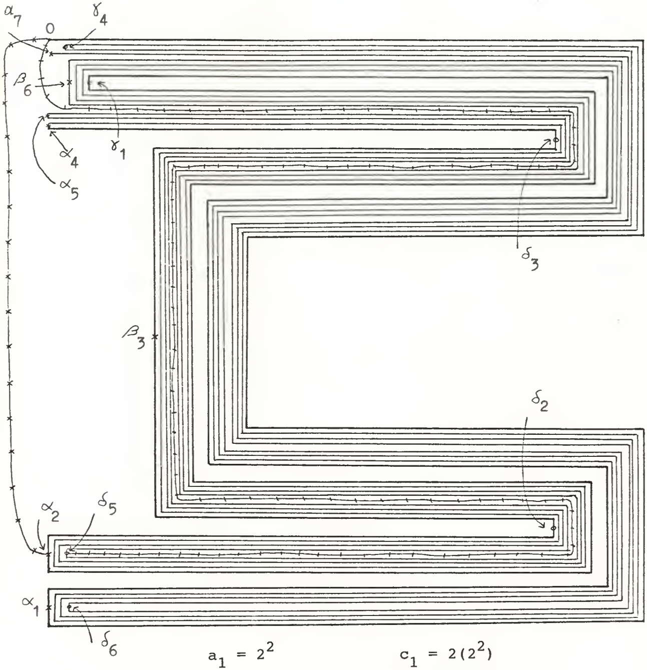


Figure 4.16



$a_1 = 2^2$ $\qquad$ $c_1 = 2(2^2)$

$a_2 = 3(2^2)$ $\qquad$ $d_2 = 3(2^2) + 2^0$

$b_3 = 3(2^2)+(2(2^0)$ $\qquad$ $d_3 = 3(2^2) + 3(2^0)$

$a_4 = 2^4$ $\qquad$ $c_4 = 2(2^4)$

$a_5 = 3(2^4)$ $\qquad$ $d_5 = 3(2^4) + 2^2$

$b_6 = 3(2^4) + 2(2^2)$ $\qquad$ $d_6 = 3(2^4) + 3(2^2)$

$a_7 = 2^6$

Figure 4.17

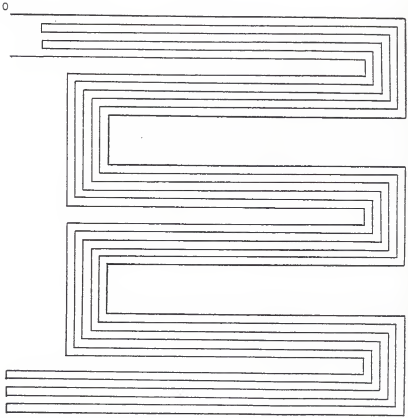


Figure 4.18

CHAPTER 5
THE PRIME END STRUCTURE OF QUOTIENT SPACES

5.1 Introduction

Suppose that X is a nonseparating continuum in S^2. Then the complement of X is a simply connected domain. Prime end theory is a way of studying the approaches to the boundary of a simply connected domain, which in this case is the boundary of X. We shall confine ourselves to the case where X has no interior, that is, is tree-like, and furthermore, where X is indecomposable.

The prime end structure of X is comprised of a homeomorphism called a C-map (defined below) $\phi:S^2-X\to\text{Ext } B$, where B is the unit circle in S^2, and the one-to-one correspondence ϕ induces between the prime ends of S^2-X and the boundary Bd B of the unit circle. Suppose that K is a proper subcontinuum of X. We can shrink K to a point in S^2 by a map $\pi:S^2\to S^2$ in such a way that $\pi(S^2)$ is homeomorphic to S^2 and π is a homeomorphism on S^2-K. Thus π on X induces an embedding of the quotient space X/K in S^2/K. Since we may regard S^2/K as S^2, it makes sense to refer to the prime end structure of $S^2/K-X/K$ as detailed by another C-map ψ: $S^2/K\to\text{Ext } B$.

In this Chapter we develop means of comparing the prime end structure of S^2-X to that of S^2K-X/K. Brechner and

Mayer [1980a] showed that equivalently embedded continua in S^2 have the same prime end structure. That is, the homeomorphism h of S^2 onto itself that sends one embedding onto the other induces a one-to-one correspondence between the prime end structures of the two embeddings; namely, the homeomormorphism h, through the C-maps, induces a homeomorphism $\overline{h}$ from Ext B onto Ext B, that preserves the classically defined kinds of prime ends. Generally the projection map will not induce a one-to-one correspondence on the respective Bd B's, nor even preserve the kinds of prime ends. However, the correspondence that is induced can be used to extend certain theorems and examples of Chapter 2, Mayer [1980b], Lewis [1981], and Krasinkiewicz [1973b]. Our applications are based upon the following Main Theorem:

5.1.1 *Theorem*

The homeomorphism $\psi\pi\phi^{-1}$: Ext B $\twoheadrightarrow$Ext B extends to a monotone map $\overline{\pi}:\overline{\text{Ext B}} \twoheadrightarrow \overline{\text{Ext B}}$.

In Section 5.2, we prove Theorem 5.1.1 and develop several corollaries. We show that an indecomposable tree-like continuum X in S^2 has a simple dense canal (Lake-of-Wada channel) in its complement if, and only if, X/K has a simple dense canal in its complement for every proper subcontinuum K. We actually prove Theorem 5.1.1 for the more general case where we have a collection H of proper subcontinua of X, each shrunk to a point.

In Section 5.3, we apply the results of Section 5.2 to several questions regarding the accessibility of composants of a tree-like indecomposable continuum in S^2. In particular, we show that no composant accessible at at most one point can contain two disjoint accessible continua (a generalization introduced by Kuratowski [1929] of the concept of an accessible point). It is a theorem of Mazurkiewicz [1929a] that an indecomposable continuum in S^2 contains at most countably many composants accessible at more than one point. We obtain the partial generalization that a tree-like indecomposable continuum contains at most countably many composants containing two disjoint accessible continua.

In Section 5.4, we apply the results of Section 5.2 to the special embedding P_s of the pseudo arc, the embedding produced by having alternate crooked chains ascending and descending. Lewis [1980] has shown that each pair of accessible points of P_s are contained in distinct composants. We show that P_s contains uncountably many nondegenerate accessible continua such that each pair are in distinct inaccessible composants. Thus P_s furnishes in a single example a continuum exhibiting many of the properties of two different examples of Krasinkiewicz [1973b].

In Chapter 2 we showed that any indecomposable chainable continuum Y can be embedded in S^2 without a simple dense canal provided that Y has at least one endpoint. In Section 5.5, we extend this theorem to the case where Y has at least one end subcontinuum, though possible no endpoints. This extension does not answer the question raised in Brechner and Mayer [1980a] as to whether every indecomposable chainable continuum can be embedded in S^2 without a simple dense canal, as there exist chainable indecomposable continua with no end subcontinua.

In Section 5.6, we show that given an indecomposable chainable continuum Y and a point p in Y, there is an embedding of Y into S^2 in which p is inaccessible. Sam Young has asked in conversation if Y can be embedded in S^2 so that p is accessible? Though we have been unable to answer Young's question, in the process several related questions arise, which we state in Section 5.6.

Throughout this paper we shall suppress reference to particular embedding functions. All spaces are metric. A double arrow indicates an onto map. For definitions and further references regarding prime ends we refer the reader to Brechner [1978]. For the relationships among prime end theory, embeddings of plane continua, and the fixed point property for nonseparating plane continua (the latter to some degree being one of the motivations for this Chapter), we refer the reader to Brechner [1979] and Brechner and Mayer [1980a, 1980b].

5.2 Prime Ends and Quotient Spaces

The classical definition of prime ends, due to Caratheodory [1913], supposes that X is the boundary of a simply connected plane domain. Ursell and Young [1951] have shown that we may take the domain to be the complement of a nondegenerate nonseparating continuum in S^2. It is that point of view which we adopt.

5.2.1 Basic Definitions in Prime End Theory.

Suppose that X is a nonseparating continuum in S^2 so that S^2-X is a simply connected domain. A crosscut of S^2-X is an open arc $Q\subset S^2-X$ such that Q is an arc and $\overline{Q}\cap X=\{a,b\}$, where a and b are the distinct endpoints of Q. An endcut of S^2-X is a half-open arc $R\subset S^2-X$ such that R is an arc and $\overline{R}\cap X=\{a\}$, where a is the other endpoint of R. A ray is a continuous one-to-one image of the half-line $[0,\infty)$. A return cut of S^2-X is an open arc $Q\subset S^2-X$ such that Q is a simple closed curve and $\overline{Q}\cap X=\{a\}$, that is, the endpoints of Q are not distinct.

A chain of crosscuts is a sequence $\{Q_i\}_{i=1}^{\infty}$ of crosscuts satisfying

(1) $\overline{Q}_i\cap\overline{Q}_j=\emptyset$, $i\neq j$.

(2) Q_i separates Q_{i-1} from Q_{i+1} in S^2-X.

(3) $\operatorname{diam}(Q_i)\to 0$ and $Q_i\to p$ in X.

We suppose that each Q_i and X miss the point at infinity. The domain cut off by Q_i, denoted U_i, is the complementary domain of $Q_i\cup X$ that does not contain the point at infinity.

A chain of return cuts fails condition (1) and instead all return cuts in the chain share a common endpoint, namely p in X. A semichain of crosscuts fails condition (1), and instead all crosscuts in the chain have exactly one endpoint in common, namely p in X. (The remaining endpoints are distinct pairwise.)

The chains of crosscuts $\{Q_i\}$ and $\{R_i\}$ are equivalent iff if $\{U_i\}$ and $\{V_i\}$ are the corresponding domains cut off, then for every j, there is a k such that $V_k \subset U_j$ and $U_k \subset V_j$. A prime end is an equivalence class of chains of crosscuts.

Given a chain of crosscuts $\{Q_i\}$ defining prime end E, the impression of E is $I(E) = \bigcap_{i=1}^{\infty} \overline{U}_i$. The set of principal points of E is $P(E)$ = the set of all points in X such that some chain of crosscuts defining E converges to x. It can be shown that $P(E) \subset I(E) \subset X$, and that both P(E) and I(E) are continua (possibly degenerate, possibly all of Bd X).

A C-map $\phi : S^2 - X \twoheadrightarrow \text{Ext } B$, where B is the unit circle in S^2, is a homeomorphism satisfying

(1) if Q is a crosscut of S^2-X, then $\phi(Q)$ is a crosscut of Bd B.

(2) the set of endpoints of images of crosscuts of S^2-X is dense in Bd B.

As Caratheodory [1913] originally developed prime end theory the map ϕ was a conformal map. Topologically it is sufficient that ϕ have the properties noted above [Ursell and Young, 1950].

Note that if $\{Q_i\}$ is a chain of crosscuts defining prime end E, then $\{\phi(Q_i)\}$ is a chain of crosscuts of S^2-B converging to a single point e in Bd B. We say that e corresponds to E. This correspondence is one-to-one [Caratheodory, 1913] and is what we mean by the prime end structure of S^2-X. If X and Y are in S^2, $\phi : S^2 - X \twoheadrightarrow \text{Ext } B$ and $\psi : S^2 - Y \twoheadrightarrow \text{Ext } B$ are C-maps, and there is a homeomorphism $h : \text{Bd } B \twoheadrightarrow \text{Bd } B$ that carries prime ends of a given kind to prime ends of the same kind, then we say the prime end structures of

S^2-X and S^2-Y are traditionally identical [Brechner and Mayer, 1980a].

Customarily there are four kinds of prime ends defined. We split three of the traditional kinds into two subkinds each, and we require that h preserve subkinds as well if the prime end structures are to be termed strongly traditionally identical.

5.2.2 Kinds of Prime Ends.

The kinds of prime ends are distinguished by which containments in the following chain are proper

$$\{p\}\subset P(E)\subset I(E)\subset \mathrm{Bd}\ X$$

where E is a prime end defined by a chain of crosscuts $\{Q_i\}$ converging to p. These kinds are:

(1) $\{p\} = P(E) = I(E) \neq \mathrm{Bd}X$

(2a) $\{p\} = P(E) \neq I(E) \neq \mathrm{Bd}X$

(2b) $\{p\} = P(E) \neq I(E) = \mathrm{Bd}X$

(3a) $\{p\} \neq P(E) = I(E) \neq \mathrm{Bd}X$

(3b) $\{p\} \neq P(E) = I(E) = \mathrm{Bd}X$

(4a) $\{p\} \neq P(E) \neq I(E) \neq \mathrm{Bd}X$

(4b) $\{p\} \neq P(E) \neq I(E) = \mathrm{Bd}X$

We refer to those prime ends whose impression is all of Bd X collectively as b-kinds and to the remainder as a-kinds. It is a theorem of Rutt [1935] that if BdX is indecomposable then X must have at least one b-kind of prime end in its complement. (See also Brechner [1978].)

Kind (3b) is particularly of interest since every embedding of any minimal tree-like counterexample to the fixed point property for nonseparating plane continua (if there is one) must have such a prime end [Brechner and Mayer, 1980a].

5.2.3 Simple Canals.

Suppose that X is a nonseparating continuum in S^2 (X may have interior). A simple canal in S^2-X is a ray $D \subset S^2-X$ satisfying

(1) $\bar{D}-D \subset \mathrm{Bd}\ X$

(2) at each point of D there is a crosscut Q of S^2-X transverse to D (separated by D with one half on each side of D.)

(3) as we choose points of D going to infinity, diam $Q \to 0$.

If further

(4) $\bar{D}-D = \mathrm{Bd}\ X$

then we say D is a simple dense canal. Roughly speaking, a simple dense canal is a ray going down a Lake-of-Wada channel in Bd X.

Brechner and Mayer [1980a] show that for any prime end E there is a simple canal D such that $\bar{D}-D = P(E)$, and conversely. The concept of a simple canal is originally due to Sieklucki [1968]. Consulting Definition 5.2.2 we see that a simple dense canal corresponds to a prime end of kind (3b). An endcut to a principal point p in X of a prime end E corresponds to E being a prime end of kind (1), (2a), or (2b); that is, p is the only principal point. We say a simple canal D is dense in K if $\bar{D}-D=K=P(E)$.

A simple canal can also be considered from the point of view of the action of a C-map ϕ. If e in Bd B corresponds to prime end E, and R is a ray in S^2-X such that $\phi(R)$ is an endcut to e, then it can be shown that $P(E)\subset\bar{R}-R\subset I(E)$. We say R defines prime end E. Now R may not be a simple canal since there may not be transverse crosscuts to X at each point of R whose diameters go to zero. (See Ursell and Young [1950].)

On the other hand, if R is a ray in S^2-X such that $\bar{R}-R\subset X$, then while $\bar{R}-R$ is always a subcontinuum of Bd X [Sieklucki, 1968], it does not follow that R defines some prime end of S^2-X. That is, $\bar{R}-R$ may not be contained in the impression of any single prime end, since $\phi(R)$ may not be an endcut to Bd B. Of course, $\phi(R)$ does converge to a subcontinuum of Bd B.

In short, any endcut to e in Bd B corresponds to a ray in S^2-X that converges to a subcontinuum of Bd X between I(E) and P(E), and any ray that converges to a subcontinuum between P(E) and I(E) corresponds to an endcut to e in Bd B. But only those rays in S^2-X with transverse crosscuts are termed simple canals, and these converge exactly to P(E).

5.2.4 The Quotient Spaces X/K and X/H.

Suppose that X is an indecomposable tree-like continuum in S^2. Then Bd X = X, and every subcontinuum of X is nonseparating. Let $\pi:S^2\twoheadrightarrow Y$ be a monotone map satisfying: $H=\{\pi^{-1}(y):\pi^{-1}(y)$ is nondegenerate$\}$ is a collection of proper subcontinua of X. Then π induces upper-semicontinuous monotone decompositions of S^2 and of X, respectively. There is thus a homeomorphism h from Y onto the induced quotient space of S^2. We denote the quotient space of S^2 by S^2/H, and we denote the induced quotient space of X by X/H. By a theorem of R. L. Moore [1925], since each element of the decomposition is nonseparating, there is a homeomorphism $g: S^2 \twoheadrightarrow S^2/H$.

Thus $g^{-1}h\pi: S^2 \twoheadrightarrow S^2$ is a monotone map, and induces an embedding of X/H into S^2. For simplicity, we will regard our monotone map π as a map from $S^2 \twoheadrightarrow S^2/H$, suppressing reference to H, and will identify S^2/H with S^2, suppressing reference to g. We regard π as inducing an embedding of X/H into S^2/H. Though we identify S^2/H with S^2, we will continue to refer to the image of π as S^2/H in order to distinguish between the simply connected domains S^2-X and $S^2/H-X/H$. In effect, π shrinks each proper subcontinuum K of X in H to a point in S^2/H. If $H = \{K\}$, we have the situation discussed in Section 5.1. For any subcontinuum M of S^2, we will denote $\pi(M)$ by M/H (or M/K, if $H = \{K\}$.)

We note that on $S^2-\overline{\bigcup H}$, π is a homeomorphism. This follows from the fact that π is one-to-one and continuous on $S^2-\overline{\bigcup H}$, and every point of $S^2-\overline{\bigcup H}$ lies in an open subset of $S^2-\overline{\bigcup H}$ with compact closure in $S^2-\overline{\bigcup H}$.

Let $\phi: S^2-X \twoheadrightarrow \text{Ext } B$ and $\psi: S^2/H-X/H \twoheadrightarrow \text{Ext } B$ be C-maps. We shall show that $\psi\pi\phi^{-1}$: Ext $B \twoheadrightarrow$ Ext B extends to a monotone map $\bar{\pi}: \overline{\text{Ext } B} \twoheadrightarrow \overline{\text{Ext } B}$, all of whose nondegenerate point inverses are in Bd B, thus proving Theorem 5.1.1. We require the following theorems as lemmas:

5.2.4.1 Theorem. Let X be an indecomposable tree-like continuum in S^2, and let $\pi:S^2 \twoheadrightarrow S^2$ be a monotone map with each nondegenerate point inverse in X. Then $\pi(X)$ is an indecomposable tree-like continuum.

Proof. That $\pi(X)$ is tree-like follows from Theorem 1 of Rosen [1958]. That X is an indecomposable continuum is fairly direct. QED

5.2.4.2 Theorem. Let X be an indecomposable tree-like continuum in S^2. Then no point of X is the single principal point of two distinct prime ends of kinds (1), (2a), or (2b) in any combination. (See Theorem 3.2 in Brechner [1978].)

5.2.4.3 Theorem. Let X be a nonseparating continuum in S^2 and let $\phi:S^2-X \twoheadrightarrow \text{Ext } B$ be a C-map. Then ϕ is uniformly continuous on the collection of crosscuts of S^2-X.

Proof. Brechner proves this in [1979]. It is a consequence of Theorem A1.4 in Ursell and Young [1951]. It does not follow that ϕ is uniformly continuous on its domain. QED

The proof of Theorem 5.1.1 then divides into proving the series of lemmas below. We remark that Theorem 5.2.4.1 allows $\pi(X)$ to be degenerate. However, our condition that H be a collection of proper subcontinua of X prevents that.

5.2.4.4 Definition of the map $\overline{\pi}$. Let E be a prime end of S^2-X defined by chain of crosscuts $\{Q_i\}$. Suppose e in Bd B corresponds to E under ϕ. By Lemma 5.2.4.5 below, the sequence $\{\pi(Q_i)\}$ determines a unique prime end E' of

$S^2/H-X/H$, which therefore corresponds to a unique point e' in Bd B under ψ.

5.2.4.5 Lemma. The sequence of open arcs $\{\pi(Q_i)\}$ in $S^2/H-X/H$ determines a unique prime end E' of $S^2/H-X/H$.

Proof. Suppose that $\{Q_i\}$ defining E converges to a point p in X. Recall that any subsequence $\{Q_j\}$ of $\{Q_i\}$ is an equivalent chain, and so determines the same prime end E. Our proof divides into establishing the following three claims:

A. There is a subchain $\{Q_j\}$ of $\{Q_i\}$ satisfying exactly one of the following conditions:

(1) $\{\pi(Q_j)\}$ is a chain of crosscuts of $S^2/H-X/H$.

(2) $\{\pi(Q_j)\}$ is a chain of return cuts of $S^2/H-X/H$.

(3) $\{\pi(Q_j)\}$ is a semichain of crosscuts of $S^2/H-X/H$.

B. In each case in part A, $\{\pi(Q_j)\}$ determines a prime end E' of $S^2/H-X/H$.

C. If $\{Q_k\}$ is any other subchain of $\{Q_i\}$, then $\{\pi(Q_k)\}$ also determines prime end E.

Thus $\{\pi(Q_i)\}$ determines a unique prime end E' of $S^2/H-X/H$. Figure 5.1 illustrates some of the elements mentioned in this proof.

A. Since π is a homeomorphism on $S^2-\bigcup H$, π preserves the separation properties of the Q_i's and the towering of the domains $\{U_i\}$ cut off by the Q_i's. That is, $\pi(Q_i)$ separates $\pi(Q_{i-1})$ from $\pi(Q_{i+1})$ in $S^2/H-X/H$, and $\pi(U_i)\supset\pi(U_{i+1})$. For a fixed crosscut Q of S^2-X, exactly one of the following conditions must obtain:

(i) neither endpoint lies in any K in H.

(ii) both endpoints lie in the same K in H.

(iii) exactly one endpoint lies in some K in H.

(iv) both endpoints lie in H, but in distinct elements K_1 and K_2 in H.

Considering the sequence $\{Q_i\}$, we observe that if there is no K in H that contains at least one of the endpoints of infinitely many of the Q_i's, then we can choose a subsequence of $\{Q_i\}$ that satisfies condition (1). Suppose that there is a K in H that contains at least one of the endpoints of infinitely many of the Q_i's. If K contains both endpoints of infinitely many crosscuts, we can choose a subsequence satisfying condition (2). Otherwise, we can choose a subsequence satisfying condition (3).

We note that it is not possible for there to be two distinct elements K_1 and K_2 of H, each of which contains an endpoint of infinitely many Q_i's, for this would imply that $\lim Q_i = \pi(p)$ is in $K_1 \cap K_2$, but $K_1 \cap K_2 = \emptyset$.

B. In each case above, if $\{Q_j\}$ is the subsequence chosen, then $\{\pi(Q_j)\}$ determines a prime end E' of $S^2/H-X/H$ as follows. If $\{\pi(Q_j)\}$ meets condition (1), then as a chain of crosscuts of $S^2/H-X/H$, $\{\pi(Q_j)\}$ determines a prime end E'. If $\{\pi(Q_j)\}$ meets condition (2), then as a chain of return cuts, each $\pi(Q_j)$ contains an endcut to the single point $\pi(p)$ in X/H. Since X/H is indecomposable by Theorem 5.2.4.1, we may apply Theorem 5.2.4.2, and the fact that an endcut to a point of X/H determines a prime end of kind (1), (2a), or (2b), to conclude that each return out determines one and the same prime end E'. The same argument applies if $\{\pi(Q_j)\}$ meets condition (3), since then each crosscut in the semichain has $\pi(p)$ as an endpoint, and so contains an endcut to $\pi(p)$.

C. We claim that if $\{\pi(Q_j)\}$ satisfies one of conditions (1), (2), or (3), then it is not possible to find a subsequence $\{Q_k\}$ of $\{Q_i\}$ such that $\{\pi(Q_k)\}$ satisfies a condition different from $\{\pi(Q_j)\}$. The proof of this claim rests upon the fact that X/H is indecomposable, implying that $\pi(p)$ is not a cut point of X/H. To establish the connection, we make two observations:

(a) Let $\pi(Q_r)$ be a return cut to $\pi(p)$, with $\pi(U_r)$ being the domain cut off, such that $\pi(U_r)$ does not contain X/H. Since $\pi(p)$ is not a cut point, $\overline{\pi(U_r)}\cap X/H = \pi(p)$. Hence, if $s > r$, then $\pi(U_r)\supset\pi(U_s)$, from which it follows that $\pi(Q_s)$ is also a return cut to $\pi(p)$.

(b) For $r < t$, let $\pi(Q_r)$ and $\pi(Q_t)$ be crosscuts with common endpoint $\pi(p)$. Suppose that $r < s < t$. Then $\pi(Q_s)$ separates $\pi(Q_r)$ from $\pi(Q_t)$ in $\pi(U_r)$. Let A be an open arc in $\pi(U_r) - \overline{\pi(U_t)}$ with endpoints in $\pi(Q_r)$ and $\pi(Q_t)$. Then A meets $\pi(Q_s)$ and separates one endpoint of $\pi(Q_s)$ from the other in $\pi(U_r) - \overline{\pi(U_t)}$. Note that $\pi(Q_r)\cup A\cup\pi(Q_t)$ contains a return out Q to $\pi(p)$. Hence by (a) the domain $U\subset(U_r)-\overline{\pi(U_t)}$ cut off by Q has the property that $\bar{U}\cap X/H = \pi(p)$. Since one endpoint of $\pi(Q_s)$ lies in U, one endpoint of $\pi(Q_s)$ must be $\pi(p)$.

It now follows that if $\{\pi(Q_j)\}$ and $\{\pi(Q_k)\}$ were to satisfy different conditions among (1), (2), and (3) then we would contradict either (a) or (b). Thus $\{\pi(Q_j)\}$ and $\{\pi(Q_k)\}$ satisfy the same condition. If both satisfy (1), then they are equivalent chains, and so determine the same prime end. If both satisfy (2) or both (3), then Theorem 5.2.4.2 applies to the endcuts contained in the return cuts or crosscuts with common endpoint $\pi(p)$. Hence again both determine the same prime end. QED

5.2.4.6 *Lemma*. The map $\overline{\pi}$ is uniformly continuous.

Proof. Though ϕ and π are not uniformly continuous on their domains, they are uniformly continuous *on the collections of crosscuts* of S^2-X and $S^2/H-X/H$, respectively, by Theorem 5.2.4.3. That is, images of small crosscuts remain small under ϕ or ψ. This observation, and the suggestion of the following argument, are due to Brechner.

We can find a simple closed curve C in S^2-X with the following properties:

(1) C is close to X.

(2) $\phi(C)$ is close to Bd B in Ext B.

(3) There is a collection of endcuts of S^2-X from points on C to X such that:

(a) The endcuts have disjoint closures, pairwise.

(b) The crosscut formed by two consecutive enducts from C and the segment of C, not containing an endpoint of any other endcut between them is small.

(See Section 5 of Sieklucki [1968].)

By the uniform continuity of ϕ on the collection of crosscuts, the image crosscuts in Ext B will be small.

Because Bd B is a circle, the closure of the domain cut off by each image crosscut in Ext B will also be small. (It is worth noting that this is not generally the case for the domains cut off by the crosscuts defined in (3a) above.) Figure 5.2 illustrates a typical selection of C and the endcuts from C to Bd B for the case where H = {K}.

Now $\pi(C)$ will be close to X/H, and the uniform continuity of π guarantees that the crosscuts formed by the images of consecutive endcuts and the segment of $\pi(C)$ between them will be small. Some of these image endcuts may now terminate at the same point in X/H, as is illustrated in Figure 5.2, if more than one have an endpoint in the same K in H. In that case, image endcuts under ψ will terminate at the same point of Bd B. However, by the uniform continuity of ψ on crosscuts (and return cuts), the image crosscuts in Ext B under ψ will still be small, and $\psi\pi(C)$ will be close to Bd B. The closures of the domains cut off by these image crosscuts in Ext B will be small, because Bd B is a circle. We note, however, that the closure of some of these domains may intersect Bd B in only a single point.

We thus see that for a close neighborhood of Bd B, bounded by ϕ (C), partitioned into small domains by endcuts from $\phi(C)$ to Bd B, that the image under $\bar{\pi}$ is a close neighborhood of Bd B, bounded by $\bar{\pi}\phi(C)$, partitioned into small image domains by image endcuts from $\bar{\pi}\phi$ (C) to Bd B. Hence $\bar{\pi}$ is uniformly continuous on $\overline{\text{Ext B}}$. QED

5.2.4.7 Lemma. The map $\bar{\pi}$ is monotone, with all nondegenerate point inverses in Bd B.

Proof. Since $\overline{\pi}$ is a homeomorphism on Ext B $\twoheadrightarrow$Ext B, we need only show that if e' is in Bd B (in the prime end structure of S^2/H-X/H), then $\overline{\pi}^{-1}(e')$ is connected. Suppose that e and f are distinct points of Bd B (in the prime end structure of S^2-X) such that both are in $\overline{\pi}^{-1}(e')$. Let P and Q be crosscuts of Ext B whose endpoints are e and f and such that $\overline{P \cup Q}$ separates Int B from the point at infinity. This situation is illustrated in Figure 5.3. Since $\overline{\pi}$ (e) = e' = $\overline{\pi}$(f), both $\overline{\pi}$(P) and $\overline{\pi}$(Q) are return cuts of S^2-B with endpoint e'. Let U and V be the domains cut off by P and Q, respectively. Without loss of generality, we may assume that $\overline{\overline{\pi}(U)} \cap$Bd B = e'. Then the facts that $\overline{\pi}$is a homeomorphism on Ext B, and continuous at Bd B, require that the interval [e,f] in $\overline{U} \cap$Bd B must go to e' under $\overline{\pi}$. Therefore, $\overline{\pi}^{-1}(e')$ is connected. QED

Lemmas 5.2.4.5 - 5.2.4.7 combined prove Theorem 5.1.1. In the following theorems, we determine the action of $\overline{\pi}$ on the impression and set of principal points of a prime end of S^2-X.

5.2.4.8 Theorem. Let E be a prime end of S^2-X corresponding to e in Bd B with $\overline{\pi}(e)$ = e', and e' corresponding to prime end E' of S^2/H-X/H. Then

(1) I(E)/H$\subset$I(E'), and

(2) P(E)/H = P(E').

Proof. To observe that (1) is the case, let R be a ray in S^2-X converging to I(E) such that ϕ(R) is an endcut to E in Bd B. Then $\psi\pi(R) = \overline{\pi}\phi(R)$ is an endcut to e' in Bd B. Now π(R) is a ray converging to a subcontinuum N = I(E)/H of

X/H. But as the inverse image of an endcut to Bd B, $\pi(R)$ converges to a subcontinuum of $I(E')$. Hence $I(E)/H=N\subset I(E')$.

Conclusion (2) follows from showing containment both ways. Suppose that D is a simple canal (see Section 5.2.3) in S^2-X converging to $P(E)$. Then $\phi(D)$ is an endcut to e in Bd B. Hence $\bar{\pi}\phi(D)$ is an endcut to e' in Bd B. But then the inverse image $\pi(D)$ is a ray converging to a subcontinuum N of X/H such that $N\supset P(E')$, and $N = P(E)/H$. Hence $P(E)/H = N\supset P(E')$.

Suppose p is a principal point of E; that is, there is a chain of crosscuts $\{Q_i\}$ defining E such that $Q_i\to p$. We claim $\pi(p)$ is in $P(E')$; that is, $\pi(p)$ is a principal point of E'. If $\{\pi(Q_j)\}$ is a chain of crosscuts of $S^2/H-X/H$ for some subsequence $\{Q_j\}$ of $\{Q_i\}$, then $\pi(Q_j)\to\pi(p)$, so $\pi(p)$ is in $P(E')$. If there is no such subsequence, then the proof of Lemma 5.2.4.5 shows that $\pi(p)$ is the single principal point of $P(E')$.

Thus $P(E)/H = \pi(P(E))\subset P(E')$. QED

5.2.4.9 Remark. We note that Theorem 5.2.4.8 does not assert that $I(E)/H = I(E')$ for $\bar{\pi}$-corresponding prime ends E and E'. Indeed, the containment may be proper. Figure 5.4 illustrates a tree-like indecomposable continuum T (shown by drawing the initial part of one composant). The chains of crosscuts $\{P_i\}$ and $\{Q_i\}$ define prime ends E and F, respectively, each of kind (2b). The chain $\{R_i\}$ defines a prime end G of kind (1). Suppose that K is the U-shaped arc $[p,q,r,s,t,u,v]$. Note that $\{p\} = P(E)$, $\{v\} = P(F)$, and $\{s\} = P(G)$, while $I(E) = T = I(F)$ and $I(G) = \{s\}$. If we shrink

K to a point, then Figure 5.4 shows the resulting embedding of T/K. Observe that map $\overline{\pi}$ carries the points e, f, and g Bd B corresponding to E, F, and G all to a single point e', corresponding to a single prime end E' of $S^2/K-T/K$. But $I(E')=T/K$. Hence, in particular, $I(G)/K \neq I(G') = I(E')$. In Theorem 5.2.4.10 below we indicate the condition sufficient for $I(E)/H$ to be $I(E')$ for $\overline{\pi}$-corresponding prime ends E and E'.

5.2.4.10 Theorem. Let E be a prime end of S^2-X corresponding to e in Bd B, $\overline{\pi}(e) = e'$, and e' corresponding to prime end E' of $S^2/H-X/H$. Suppose for some chain of crosscuts $\{Q_i\}$ defining E, $\{\pi(Q_i)\}$ is a chain of crosscuts of $S^2/H-X/H$. Then $I(E)/H = I(E')$.

Proof. From the proof of Lemma 5.2.4.5, it follows that $\{\pi(Q_i)\}$ defines E'. Since π is a homeomorphism on $S^2-\overline{\cup H}$ we have:

$$I(E') = \bigcap_{i=1}^{\infty}\overline{\pi(U_i)} = \bigcap_{i=1}^{\infty}\pi(\overline{U}_i)=\pi\left(\bigcap_{i=1}^{\infty}\overline{U}_i\right) = I(E)/H$$

where the U_i's are the domains cut off by the Q_i's. QED

5.2.4.11 Remark. We have proved the theorems in Section 5.2.4 for the case in which the collection H of proper subcontinua of X may contain many elements. In most of the applications that follow, we will be concerned with the case where only a single proper subcontinuum of X is shrunk to a point. We have also limited ourselves to the case where X is tree-like and indecomposable. These limitations do not seem inherent in the material.

It is reasonable to conjecture that the induced correspondence $\overline{\pi}$ would be a well-defined monotone map when X is separating, provided we restrict ourselves to a given complementary domain of S^2-X, and shrink out only nonseparating subcontinua. It also seems reasonable that $\overline{\pi}$ would exist if X were decomposable, though in that case a considerably different proof of Lemma 5.2.4.5 would have to be found, as we made use of indecomposability repeatedly in our proof.

5.2.5 The Action of $\overline{\pi}$ on Kinds of Prime Ends.

Theorem 5.2.4.8 places certain limitations on the kinds of prime ends that can correspond under $\overline{\pi}$. That is, if E is a prime end of S^2-X which $\overline{\pi}$-corresponds to E', a prime end of S^2/H-X/H, then the kind of E determines the range of kinds that E' can fall into. Theorem 5.2.5.1 below states the "theoretical" range precisely. We say "theoretical" because we do not have examples for the cases in square brackets. The examples for the remaining cases are below.

5.2.5.1 Theorem. The following diagram indicates the possible range of $\bar{\pi}$-corresponding kinds of prime ends:

Kind of E:	Possible Kinds of E':
(1)	(1), (2a), (2b)
(2a)	(1), (2a), (2b)
(2b)	(2b)
(3a)	(1), (2a), (2b), (3a), [4a], [4b]
(3b)	(3b)
(4a)	(1), (2a), (2b), (3a), (4a), [4b]
(4b)	(2b), (4b)

Proof. If E is of kind (1), (2a), or (2b), then P(E) is trivial. Hence P(E') is also trivial, since P(E') = P(E)/H. But because I(E') may exceed I(E)/H, the possibilities are as indicated. In case I(E) = X, I(E') X/H, thus a prime end of kind (2b) can only go to a prime end of kind (2b) under $\bar{\pi}$. If E is of kind (3a) or (4a), then P(E) is proper, so necessarily P(E') is proper, though possibly has shrunk to degeneracy. Hence (3b) is excluded for E', but we might have any of the others. If E is of kind (3b), then P(E') = I(E') = X/H, so E' is also of kind (3b). If E is of kind (4b), then P(E') is proper, and possibly degenerate. Since I(E')$\supset$X/H, E' is either of kind (2b) or (4b). QED

5.2.5.2 Theorem. If for prime end E of S^2-X, $P(E)\subset K$ with K in H, then the following diagram represents the range of kinds of prime end that E' can fall into:

Kind of E:	Possible Kinds of E':
(1), (2a), (3a), (4a)	(1), (2a), (2b)
(2b), (4b)	(2b)

Proof. If $P(E)\subset K$ in H, then $P(E')$ is degenerate. This observation and Theorem 5.2.5.1 imply the above range of correspondences. QED

5.2.5.3 Theorem. On prime ends of kind (3b), the correspondence induced by $\overline{\pi}$ is one-to-one.

Proof. We observe first that E is of kind (3b) iff E' is of kind (3b), by Theorem 5.2.5.1. Suppose E and F are distinct prime ends of kind (3b) in S^2-X. Let e, f be the (distinct) points in Bd B corresponding to E and F with $e'=\overline{\pi}(e)$ and $f'=\pi(f)$. Let E' and F' be the corresponding prime ends of $S^2/H-X/H$. We claim E' and F' are distinct prime ends.

Let $\{Q_i\}$ and $\{R_i\}$ be chains of crosscuts defining E and F respectively, with $\{U_i\}$ and $\{V_i\}$ the domains cut off. Since E and F are distinct prime ends, we may assume that no Q_i meets any V_j and no R_i meets any U_j. By Theorem 5.2.5.1 E' and F' are of kind (3b). Hence $\{Q_i\}$ does not contain a subsequence $\{Q_j\}$ such that $\{\pi(Q_j)\}$ is a chain of return cuts or semichain of crosscuts. So we may assume $\{\pi(Q_i)\}$ is a chain of crosscuts of $S^2/H-X/H$. Similarly, we may assume $\{\pi(R_i)\}$ is a chain of crosscuts. Since π is a homeomorphism on $S^2-\overline{\bigcup H}$, no $\pi(Q_i)$ meets any $\pi(V_j)$ and no $\pi(R_i)$ meets any $\pi(U_j)$. Hence E' and F' are distinct prime ends. QED

5.2.5.4 Corollary. Let X be an indecomposable tree-like continuum in S^2. Let K be a proper subcontinuum of X. Then S^2-X contains exactly as many simple dense canals in its complement as does $S^2/K-X/K$.

5.2.5.5 Remark. Actually Corollary 5.2.5.4 clearly also holds when H is a collection of proper subcontinua that we shrink to points. What it reveals is that our shrinking process preserves simple dense canals, or the absence thereof. We shall apply this fact in Section 5.5.

Also note, that if $P(E)\subset K$ in H, then E becomes a prime end E' with exactly one principal point, and E' is defined by an endcut to the principal point $P(E)/H$ in X/H. We shall apply this fact in Section 5.3.

5.2.6 Examples.

We show in the following examples, that the range of kinds of prime ends of $S^2/H-X/H$ to which a given prime end of S^2-X can correspond under $\overline{\pi}$ is as indicated in Theorem 5.2.5.1. Consider first the continuum T of Figure 5.4. We have already shown that $\overline{\pi}$ can carry a prime end of kind (2b) to a prime end of kind (2b), in Remark 5.2.4.9. We produce further examples by making certain modifications of T in the neighborhood of arc [p,s,v].

Let M_i (i=1,2,3,4) be the four continua depicted in Figure 5.5. If we simply replace arc [p,s,v] of T by each of M_1 through M_4 in turn, in the obvious way, the resulting continua are each decomposable. To produce indecomposable examples, we use M_1 through M_4 as templates for modifying T in the neighborhood of arc [p,s,v]. Imagine that we have temporarily replaced arc [p,s,v] by M_1 in T. Supposing that T is elastic, place your pencil just above T near the point where the first sticker of M_1 joins [p,q]. Now with the pencil point push all of the composants of T in a small neighborhood of the pencil point downward to the end of the first sticker, stretching, but not breaking, the small subarcs in the line-like composants. Do this for each sticker in turn. The resulting continuum, T_1, mimics the structure of $T \cup M_1$, but is in indecomposable. (Make sure you have removed the temporary template M_1.) Portions of T that formerly did not converge to points of arc [q,u] now do, so T_1 is not homeomorphic to T. In a similar manner construct T_i (i=2,3,4) mimicking the structure of $T \cup M_i$, but making it indecomposable. Figure 5.6 illustrates T_i as the intersection of a defining sequence of tree covers.

We can now exemplify several of the other possible correspondences stated in Theorem 5.2.5.1.

5.2.6.1 Example. (2a)→(1): A ray D converging to {s} in T_2 determines a prime end E of kind (2a), with P(E) = {s}, and I(E) = [q,u]. If we shrink [q,u] to a point in T_2, then $T_2/[q,u] = T_3$, and D becomes an endcut to s in T_3, determing a prime end E' of kind (1), since I(E') = {s} = P(E').

5.2.6.2 Example. (2a)→(2b): In our strethcing construction above, arcs $[p,q]$ and $[u,v]$ of T have become rays in T_i, converging to some subcontinuum of arc $[q,r,s,t,u]$. We will designate these rays as $[p,q)$ and $(u,v]$. In T_2, ray $[p,q)$ converges to arc $[q,s]$ and ray $(u,v]$ converges to arc $[s,u]$. Suppose we shrink $K = [p,q) \cup [q,s,u] \cup (u,v]$ to a point. Then T_2/K is homeomorphic to the Knaster U-continuum (see Definition 5.6.4.1), with the point K/K to which K has shrunk being the single endpoint. An endcut D in $S^2 - T_2$ to s becomes an endcut to the single endpoint of the Knaster U-continuum, and defines a prime end of kind (2b). But in T_2, D defines a prime end E of kind (2a), since $I(E) = [q,s,u]$ and $P(E) = \{s\}$.

5.2.6.3 Example. (3a)→(1): A ray D in $S^2 - T_4$ converging to $[q,s,u]$ determines a prime end E of kind (3a) since $I(E) = P(E) = [q,s,u]$. If we shrink $[q,s,u]$ to a point, then $T_4/[q,s,u] = T_3$, and D becomes an endcut to s in T_3, determining a prime end of kind (1).

5.2.6.4 Example. (3a)→(2b): Again let D in $S^2 - T_4$ be a ray converging to $[q,s,u]$. Now shrink $[p,q) \cup [q,s,u] \cup (u,v] = K$ to a point. Then T_4/K is homeomorphic to the Knaster U-continuum, and D becomes an endcut to the single endpoint K/K determining a prime end of kind (2b).

5.2.6.5 Example. (4a)→(1): A ray D in $S^2 - T_1$ converging to $[r,u]$ determines a prime end E with $P(E) = [r,u]$ and $I(E) = [q,u]$. Hence E is of kind (4a). If we shrink $[q,u]$ to a point, then $T_1/[q,u] = T_3$, and D becomes a ray converging to s determining a prime end of kind (1).

5.2.6.6 *Example*. (4a)→(3a): In the situation of Example 5.2.6.5, shrink $[q,r]$ to a point. Then D becomes a ray converging to $[q,u]$ in $T_4 = T_1/[q,r]$, with every point of the impression a principal point.

5.2.6.7 *Example*. (4a)→(2a): Let D in S^2-T_1 be a ray converging to $[r,u]$, and so defining a prime end E of kind (4a). If we shrink $[r,u]$ to a point, then $T_1/[r,u] = T_2/[s,u]$, and D becomes an endcut to s in $T_2/[s,u]$ determining a prime end of kind (2a).

5.2.6.8 *Example*. (4a)→(2b): In the situation of Example 5.2.6.7, shrink all of $K = [p,q) \cup [q,u] \cup (u,v]$ to a point. Thus T_1/K = Knaster U-continuum, and ray D becomes an endcut to the single endpoint K/K of the U-continuum. The image of D determines a prime end of kind (2b).

The identity map will serve to exemplify correspondences between prime ends of the same kind, though more complicated examples can be constructed by the reader using the continua T_i and shrinking out only *part* of a set of principal points. The following examples for (1)→(2a) and (3a)→(2a) require constructing some new continua.

5.2.6.9 *Example*. (1)→(2a): Let X be the continuum in S^2 defined by the sequence $\{D_i\}$ of chain covers suggested by Figure 5.7. Note that $S = \bigcap_{i=1}^{\infty} D_i(1,a_i)^*$ is a proper subcontinuum of X; in fact, S is a Knaster S-continuum. Note that $A = \bigcap_{i=1}^{\infty} D_i(a_i,b_i)^*$ is an arc. Let $a = \bigcap_{i=1}^{\infty} D_i(a_i)$, $b = \bigcap_{i=1}^{\infty} D_i(b_i)$, and $p = \bigcap_{i=1}^{\infty} D_i(1)$. It can be shown that p is the only endpoint of X, that p and a are the endpoints of S, and that a and b are the endpoints of A. In S^2-X, and endcut R_1 (hatched line in Figure 5.7) to point a defines a prime end E of kind (2a) with I(E)=S; an endcut R_2 (x-ed line in Figure 5.7) to b defines a prime end F of kind (1). If we shrink arc A to a point in S^2, by monotone map π, then $\pi(R_1)$ and $\pi(R_2)$ are endcuts to the same point $\pi(a) = \pi(b) = \pi(A)$ in $S^2/A-X/A$. Both $\pi(R_1)$ and $\pi(R_2)$ define a prime end E' of kind (2a) with $P(E') = \pi(A)$ and $I(E') = S/A = S$. Hence F of kind (1) is carried to E' of kind (2a).

5.2.6.10 Example. (3a)→(2a): We modify Example 5.2.6.9, as shown in Figure 5.8, in order to introduce a prime end of kind (3a) to the complement. Let X be the continuum defined by the sequence $\{D_i\}$ of chain covers suggested by Figure 5.8. Note that $S = \bigcap_{i=1}^{\infty} D_i(1,a_i)^*$ is an S-continuum properly contained in X. Note also that $U = \bigcap_{i=1}^{\infty} D_i(a_i,b_i)^*$ is a U-continuum. In fact, $S \cup U$ is the union of S and U at a common endpoint $a = \bigcap_{i=1}^{\infty} D_i(a_i)$. (Point a is not an endpoint of X; the intersection of first links is the only endpoint of X.) An endcut R_1 (x-ed line in Figure 5.8) to point a defines a prime end E of kind (2a) in S^2-X with $I(E) = S$. A simple canal R_2 converging to U, (hatched line in Figure 5.8) defines a prime end F of kind (3a) with $P(F) = U = I(F)$. If we shrink U to a point in S^2 by monotone map π, then $\pi(R_1)$ and $\pi(R_2)$ are endcuts to the same point $\pi(a) = \pi(U)$ in X/U. In $S^2/U-X/U$, both $\pi(R_1)$ and $\pi(R_2)$ define a prime end E' of kind (2a) with $P(E') = \pi(U) = \pi(a)$, and $I(E') = S/U = S$. Thus F of kind (3a) is carried to E' of kind (2a).

Note that R_2 is a simple canal of S^2-X *dense in K*, but R_2 is *not* a simple dense canal of S^2-K. The crosscuts to X transverse to R_2 do not have both endpoints in K.. In effect, R_2 "wraps around" K. There is no s-d-c in S^2-K.

5.2.6.11 *Remark*. For correspondences (3a)→(4a), (3a)→(4b), and (4a)→(4b), we have neither examples, nor proof that such correspondences cannot occur. Example 5.4.2 to follow will illustrate (4b)→(2b). This latter class of correspondences will occupy our attention in most of the next two sections.

5.3 Accessibility

We use the results of the previous section to extend theorems concerning the accessibility of points, composants and subcontinua of an indecomposable tree-like continuum X in S^2. Generally, we will be shrinking only a single proper subcontinuum K of X to a point.

5.3.1 Definitions.

A point p in X is *accessible* iff there is a ray R in S^2-X such that $\overline{R}-R = \{p\}$. We have previously observed that R is then an endcut determining a prime end E of S^2-X with $P(E) = \{p\}$.

A composant C of X is *accessible* iff C contains at least one accessible point. If point p in C is accessible, we say C is *accessible at p*.

A proper subcontinuum K of X is an *accessible continuum* iff there is a ray R in S^2-X such that $\overline{R}-R \subset K$. Clearly, an accessible point is a (degenerate) accessible continuum;

however, an accessible continuum may contain no accessible points, as examples to follow will show.

A composant C of X is a K-composant (or continuum-wise accessible composant) iff C contains an accessible continuum. The concept of a K-composant is due to Krasinkiewicz [1972, 1973a, 1973b], though the concept was first utilized by Kuratowski [1929], in that the latter defined "accessible continuum."

5.3.2 Remark.

In the aforementioned papers, Krasinkiewicz defines an accessible continuum of X as subcontinuum K of X such that there is a continuum L in S^2 with $L\cap X = K$ and $L-X\neq\emptyset$. We show in Theorem 5.3.4 that for indecomposable tree-like continua, this definition and Definition 5.3.1 are equivalent.

If X contains an accessible continuum K in composant C of X, then X generally contains more than one accessible continuum in the same composant, but in a certain sense, the additional accessible continua may not be all that different from K. Suppose that $L\cap X = K$ and $L-X\neq\emptyset$. Let M be a subcontinuum of X with $M-K\neq\emptyset\neq M\cap K$. Then $(L\cup M)\cap X = K\cup M$ and $(L\cup M)-X\neq\emptyset$, though $(L\cup M)-X = L-X$. We have not changed L-X, but have constructed another accessible continuum $K\cup M$ in the same composant.

Alternately, let $L_1\cap X = K_1$ and $L_2\cap X = K_2$ with $L_1-X\neq\emptyset$ and $L_2-X=\emptyset$ and $K_1\cap K_2\neq\emptyset$. Again K_1 and K_2 are not really different ways of "reaching" composant C from the complement. For both $L_1 \cup K_2$ and $L_2 \cup K_1$ reach the accessible continuum $K_1 \cup K_2$. Of particular interest to us is under what conditions there can be two distinctly different ways of reaching composant C of X, that is, when $L_1 \cap X$ and $L_2 \cap X$ are disjoint accessible continua. Theorem 5.3.7 tells us that C must then be accessible at more than one point. The interesting fact here is that if C is continuum-wise accessible in two disjoint continua, then C is point-wise accessible, and at uncountably many points (more than one implies uncountably many.)

Krasinkiewicz [1973b] defines a composant C of X to be an external composant iff there is a continuum L in S^2 such that $L \cap C \neq \emptyset$, $L \not\subset C$, and L does not intersect all composants of X.

5.3.3 Definition

A point p in X is accessible with respect to prime end E iff there is an endcut to p in S^2-X which determines E. A point p accessible with respect to a prime end E is accessible, and conversely, an accessible point is accessible with respect to some prime end E. However, a point p may be accessible with respect to prime end E, and lie in P(F) for a different prime end F without being accessible with respect to prime end F. (See Example 2.5.2 or Figure 2.3(b) in Chapter 2.)

5.3.3.1 Theorem. Let X be a tree-like continuum in S^2. If E is a prime end of S^2-X of an a-kind, then composant C of X containing P(E) is accessible at more than one point. (See Theorem 3.5 of Brechner [1978].)

5.3.3.2 Theorem. Let X be an indecomposable tree-like continuum in S^2. Then no point p in X is accessible with respect to two distinct prime ends. (Theorem 3.2 of Brechner [1978].)

5.3.4 Theorem

Suppose that X is a tree-like indecomposable continuum in S^2, and that K is a proper subcontinuum of X. Then there is a continuum $L \subset S^2$ with $L \cap X = K$ and $L - X = \emptyset$ iff there is a ray $D \subset S^2 - X$ with $\bar{D} - D \subset K$.

Proof. If D is a ray in $S^2 - X$ with $\bar{D} - D \subset K$, then $\bar{D} \cup K$, is a continuum in S^2 with $(\bar{D} \cup K) \cap K = K$ and $(\bar{D} \cup K) - X \neq \emptyset$.

Conversely, suppose that L is a continuum in S^2 with $L \cap X = K$ and $L - X \neq \emptyset$. Let $\pi : S^2 \twoheadrightarrow S^2/K$ be the monotone map induced by shrinking K to a point. In S^2/K, $\pi(L) = L/K$ is a continuum such that $L/K \cap X/K = \pi(K) = k$, a single point in X/K. Because k is a cut point of $L/K \cup X/K$ in S^2, we can then construct an endcut D' in $S^2/K - X/K$ to k (that is, k is an accessible point in X/K.) Then $D = \pi^{-1}(D')$ is a ray in $S^2 - X$ such that D converges to a subcontinuum of K; that is, $\bar{D} - D \subset K$. QED

5.3.5 Theorem

Let X be an indecomposable tree-like continuum in S^2. Let E and F be distinct prime ends of kinds (2b) or (4b) in any combination. Then $P(E)$ and $P(F)$ lie in the same composant C of X only if C is accessible at more than one point.

Proof. Suppose $P(E) \cup P(F) \subset C$, and that C is accessible at at most one point. Let K be the (proper) subcontinuum of

X irreducible between P(E) and P(F). Let $\{Q_i\}$ and $\{R_i\}$ be chains of crosscuts defining E and F, respectively. Since E and F are distinct prime ends, these chains are not equivalent. Hence we may assume that if $\{U_i\}$ and $\{V_i\}$ are the domains cut off by $\{Q_i\}$ and $\{R_i\}$, respectively, then no Q_i meets any V_j and no R_i meets any U_j.

Since C is accessible at at most one point, the endpoints of each Q_i and each R_i may be assumed to lie outside K. Let $\pi : S^2 \twoheadrightarrow S^2/K$ be the monotone map induced by shrinking K to a point. Then $\{\pi(Q_i)\}$ and $\{\pi(R_i)\}$ are chains of crosscuts to X/K in $S^2/K-X/K$. Thus $\{\pi(Q_i)$ and $\{\pi(R_i)\}$ define distinct prime ends E' and F' of $S^2/K-X/K$, respectively. Now $P(E') = P(E)/K = \pi(K) = P(F)/K = P(F')$ by Theorem 2.4.8. Since $\pi(K)$ is a single point, E' and F' are of kind (2b) with the same principal point $\pi(K)$. Since X/K is indecomposable, this contradicts Theorem 5.2.4.2. Hence C must be accessible at more than one point. QED

5.3.6 Theorem

Let X be a tree-like continuum in S^2 and let K be an accessible subcontinuum of X. Then there is a prime end E of S^2-X such that $P(E) \subset K$.

Proof. Let $\pi : S^2 \twoheadrightarrow S^2/K$ be the monotone map induced by shrinking K to a point. Let $\overline{\pi}$ be the monotone map on the prime end structures of S^2-X and $S^2/K-X/K$, via Theorem 5.1.1. Let R be a ray in S^2-X such that $\overline{R}-R \subset K$. Then $\pi(R)$ is an endcut to the point $\pi(K)$ in X/K, and thus determines a prime end E' of $S^2/K-X/K$ with $P(E') = \pi(K)$, a single point. Let e' in Bd B correspond to E'. Since $\overline{\pi}$ is onto, there is an e in Bd B such that $\overline{\pi}(e) = e'$. Let E be the prime end of S^2-X that corresponds to e. By Theorem 5.2.4.8, we know that $P(E') = P(E)/K$. Since $P(E') = \pi(K)$, this implies $P(E)\subset K$. QED

5.3.7 Theorem

Let X be an indecomposable tree-like continuum in S^2. Let K and L be disjoint accessible continua of X that lie in the same composant C of X. Then C is accessible at more than one point.

Proof. Let E and F be prime ends of S^2-X such that $P(E)\subset K$ and $P(F)\subset L$, via Theorem 5.3.6. Since $L\cap K=\emptyset$, P(E) and P(F) are disjoint, so E and F are distinct prime ends. If either E or F are of an a-kind (including kind (1)), then C is accessible at more than one point by Theorem 5.3.3.1. Since both P(E) and P(F) are properly contained in X, neither E nor F are of kind (3b). If E and F are of kinds (2b) and (4b) in some combination, then C is accessible at more than one point by Theorem 5.3.5. Thus whatever the kinds of E and F, C is accessible at more than one point. QED

It is a theorem of Mazurkiewicz [1929b] that only countably many composants of an indecomposable plane continuum are accessible at more than one point. In the case of tree-like continua, we can now generalize Mazurkiewicz's theorem to accessible continua, as a corollary to Theorem 5.3.7.

5.3.8 Corollary

Let X be an indecomposable tree-like continuum in S^2. Then there are at most countably many composants of X which contain two disjoint accessible continua.

Proof. Suppose X contains uncountably many composants, each containing two disjoint accessible continua. Then by Theorem 5.3.7, X contains uncountably many composants accessible at more than one point. This contradicts Mazurkiewicz's theorem. QED

5.3.9. Remark

If we are correct in our conjecture in Remark 5.2.4.11 that Theorem 5.1.1 can be extended to separating indecomposable continua, then we would have a complete generalization of Mazurkiewicz's theorem to accessible continua. (Here "accessible" would have to be taken to mean "accessible with respect to some complementary domain" of X in S^2-X.)

5.4 The Pseudo Arc

In this section we apply the results of the previous section to the special embedding P_s of the pseudo arc in S^2. In fact, there are uncountably many inequivalent embeddings of the pseudo arc in S^2 (see Lewis [1980], and Smith [1980]), but the special embedding, defined below, has several interesting properties

(1) Each pair of accessible points lie in distinct composants.

(2) Each pair of nondegenerate disjoint accessible continua lie in distinct inaccessible composants, provided the continua contain no accessible points.

(3) There are uncountably many simple dense canals in S^2-P_s.

(4) There are uncountably many accessible composants.

(5) There are uncountably many nondegenerate disjoint accessible continua.

(6) There are uncountably many inaccessible K-composants.

(7) All prime ends of S^2-P_s are b-kinds.

Property (1) was demonstrated by Lewis [1980]. Property (4) follows from (1) and the fact that any plane continuum contains uncountably many accessible points. Property (1) implies that the continuum irreducible between the endpoints of any crosscut is P_s. Consequently (7) follows by the definition of the impression of a prime end. Property (3) was demonstrated in Chapter 2 and Mayer [1980b]. We prove properties (2), (5), and (6) in the theorems below.

5.4.1 Definition

The special embedding P_s of the pseudo arc in S^2 is the intersection of a defining sequence (in the sense of Ingram and Cook [1967]) $\{D_i\}$ of convex open disk chains in S^2 such that

(1) $D_i(1) \subset D_{i-1}(1)$ and $D_i(n_i) \subset D_{i-1}(n_{i-1})$.

(2) D_i is crooked in D_{i-1} with minimal bending.

(3) For even i, D_i is descending in D_{i-1}.

(4) For odd i, D_i is ascending in D_{i-1}.

Brechner [1978] first defined the special embedding and conjectured that no composant contained more than one accessible point. For the definition of crooked see Bing

[1948]. For the definitions of defining sequence, ascending, descending, minimal bending, and other useful terms for describing chains see Mayer [1980b] and Chapter 2.

5.4.2 Theorem

Let P_s be the special embedding of the pseudo arc in S^2, and let $\phi: S^2 - P_s \twoheadrightarrow \text{Ext } B$ be a C-map. Then all prime ends of $S^2 - P_s$ are of b-kinds, specifically (2b), (3b), and (4b), and the collection of points corresponding to each kind is dense and uncountable in Bd B.

Proof. We have already noted that all prime ends are b-kinds. The set of accessible points and the set of endpoints of crosscuts coincide, so prime ends of kind (2b) are dense and uncountable in Bd B.

We showed in Mayer [1980b] and Chapter 2 that there are uncountably many simple dense canals in $S^2 - P_s$ by constructing an infinite binary tree in $S^2 - P_s$ such that each branch was a s-d-c. Since we started our construction in an arbitrary link, we can show that points corresponding to prime ends of kind (3b) are dense and uncountable in Bd B. By a similar construction, detailed below, we will show that points corresponding to prime ends of kind (4b) are dense and uncountable in Bd B.

Figure 5.9 shows how we construct a ray R dense in a proper subcontinuum K of P_s in a fashion similar to the proof of Theorem 2.3.7 in Chapter 2. The ray R is so constructed that it is a simple canal dense in a nondegenerate continuum K, and so defines a prime end of kind (4b). (However, note that R is not a s-d-c of $S^2 - K$;

see Example 5.2.6.10.) Since we may start our canal in any link, we get density of the corresponding points in Bd B. By showing how to modify the ray R to become an infinite binary tree, we get uncountability of the corresponding points in Bd B.

We lose no generality by beginning R with an endcut R_1 to the boundary of a link $D_1(b_1)$ of chain D_1 in the defining sequence $\{D_i\}$ for P_s. Let $D_1(a_1,b_1)$ be a subchain of at least five links. Chain D_2 is descending in D_1, so there is a looping subchain $D_2(a_2,b_2,c_2)$, as shown in Figure 5.9, running from $D_1(b_1)$ to $D_1(a_1)$ and back to $D_1(b_1)$ with $D_2(a_2,b_2,c_2) \subset D_1(a_1,b_1)$, $D_2(a_2) \cup D_2(c_2) \subset D_1(b_1)$, and $D_2(b_2) \subset D_1(a_1)$. We extend R_1 by R_2 so that arc R_2 starts at the endpoint of R_1 in Bd $D_1(b_1)$, meets all links of $D_1(a_1,b_1)$ with minimal bending, meets no other links of D_1, and misses all links of D_2, except at its endpoint in Bd $D_2(b_2)$. Only a very few of the many bends in D_2 are shown in Figure 5.9.

Similarly, we continue with chain D_3 which is ascending in D_2. Consequently, there is a looping subchain $D_3(a_3,b_3,c_3)$, as shown in Figure 5.9, running from $D_2(b_2)$ to $D_2(c_2)$ and back to $D_2(b_2)$ without leaving the subchain $D_2(b_2,c_2)$. We extend R_2 by R_3 so that arc R_3 starts at the endpoint of R_2 in Bd $D_2(b_2)$ and proceeds in the obvious fashion between links of $D_3(a_3,b_3,c_3)$ to terminate at Bd $D_3(b_3)$. Since chains D_i alternate descending and ascending, we may continue our construction inductively, and so define a ray $R = \bigcup_{i=1}^{\infty} R_i$. Note that at each point of R there is a transverse crosscut to P_s which may be confined to a single link of the chain D_{i-1}, assuming the point lies in $R_i \subset R$. Hence R is a simple canal dense in the subcontinuum $K = \bigcap_{i=1}^{\infty} D_i(a_i,b_i,c_i)^*$. Assuming that at least one of a_1 and b_1 is not an endlink, K is proper and nondegenerate in P_s. Thus R defines a prime end of kind (4b).

Note that $\{D_i(a_i,b_i,c_i)\}$ is not a sequence of minimal covers of K. For such a minimal sequence we must select subchains $D_i(a_i,b_i)$ for odd i, and $D_i(b_i,c_i)$ for even i.

A branching procedure, illustrated at the bottom of Figures 5.9, allows us to introduce a branch to R at the i^{th} stage so that the branch goes down the center of a different loop of the i^{th} chain. Because the chains are crooked, we only have to move over one link to find a different loop, so the branch also will define a simple canal. By introducing a branch at each stage to each branch already growing, we construct an infinite binary tree in S^2-P_s, each of whose branches defines a prime end of kind (4b). That these are distinct prime ends follows from the fact that the continua to which they converge are disjoint, and consequently their sets of principal points are disjoint. QED

Since each of the simple canals in the proof of Theorem 5.4.2 converges to a proper subcontinuum of P_s of which no pair intersect, property (5) follows. The next theorem shows that each prime end of kind (4b) or (2b) must have its set of principal points in a composant different from that of any other prime end of kind (4b) or (2b).

5.4.3 Theorem

No two disjoint accessible continua of P_s lie in the same composant.

Proof. By Theorem 5.3.7, if K and L are disjoint accessible continua of P_s lying in composant C, then C is accessible at more than one point. This contradicts the fact that no composant of P_s is accessible at more than one point. QED

5.4.4 Theorem

Each prime end E of kind (4b) in S^2-P_s has its principal continuum P(E) in an inaccessible composant C of P_s, and C contains the set of principal points of no other prime end.

Proof. Since no composant of P_s is accessible at more than one point, Theorem 5.3.5 requires that for any other prime end F of kind (2b) or (4b), P(F) lie in a composant different from C. Hence C is inaccessible, and C contains P(F) for no other prime end F. QED

We observe that property (2) follows from Theorem 5.4.3, and that property (6) follows from Theorems 5.4.2 and 5.4.4.

5.4.5 Remark

Krasinkiewicz in [1973b] provides examples of plane indecomposable continua exhibiting certain of the accessibility properties he defines in that paper. In order to compare the properties of his examples to those of P_s, we will need a few more definitions, all due to Krasinkiewicz.

A point p in indecomposable continuum X in S^2 is weakly accessible from S^2-C(p), where C(p) denotes the composant of p. A composant C of X containing a weakly accessible point is said to be a weakly accessible composant. An indecomposable continuum X in S^2 is of finite type iff S^2-X has a finite number of components. Obviously tree-like continua are of finite type. Theorem 3.3 of Krasinkiewicz [1973b] states that an indecomposable continuum in S^2 of finite type has each weakly accessible composant accessible.

Example 4.1 of Krasinkiewicz [1973b] is an indecomposable continuum X in S^2 having a set of cardinality c of external inaccessible composants. Since K-composants are external composants, and each nondegenerate accessible continuum in the uncountably collection we have defined for P_s determines an inaccessible K-composant, the special embedding of the pseudo arc has this property. However, Example 4.1 has uncountably many weakly accessible inaccessible composants, which are therefore external composants but not K-composants. Necessarily Example 4.1 is not of finite type. Thus Krasinkiewicz's example is richer than P_s in certain respects.

Example 4.2 of Krasinkiewwicz [1973b] is an indecomposable continuum Y in S^2 such that the set of accessible composants is of cardinality c. From the construction Y is chainable and contains arcs as subcontinua. The pseudo arc also has the property of having uncountably many accessible composants, in the special embedding. Of course, P_s contains no arcs.

In summary, P_s combines some of the properties of the above two examples: uncountably many accessible composants and uncountably many inaccessible external composants.

5.5 Chainable Continua with End Subcontinua

In Mayer [1980b] and Chapter 2 we showed that every indecomposable chainable continuum X with at least one endpoint can be embedded in S^2 without a s-d-c. In fact, the embedding has only one prime end E, corresponding to an endcut to the endpoint, such that the impression of E is all of X. In this section we extend that theorem to a larger class of chainable continua, those with at least one end subcontinuum, defined below. Since there are chainable indecomposable continua without end subcontinua, as we show by example, we have not yet been able to extend the proof to all chainable continua.

5.5.1 Definition

Let X be a chainable indecomposable continuum. If for some proper subcontinuum K of X and for some defining sequence of chain covers $\{C_i\}$ of X, there is, for all i, a k_i with $1 \leq k_i < n_i$ such that:

(1) $C_i(1,k_i) \subset C_{i-1}(1,k_{i-1})$, and

(2) $K = \bigcap_{i=1}^{\infty} C_i(1,k_i)^*$

then K is an end subcontinuum of X.

5.5.1.1 Example. Suppose X has an endpoint p. Then p is a (degenerate) end subcontinuum of X.

5.5.1.2 Example. In Mayer [1980b] and Chapter 2 we showed that there is an indecomposable chainable continuum N with no endpoints. In fact we gave several examples of such continua. It turns out that most of our examples have end subcontinua. We reproduce one such example below. Figure 5.10 illustrates the inverse-limit-with embedding diagram for continuum N, and shows how N may be embedded in S^2 as the intersection of defining sequence $\{D_i\}$ of convex open disk chain covers. Each chain $D_i(1,p_i,n_i)$ is composed of three straight-through subchains, with $D_i(1) \cup D_i(p_i) \subset D_{i-1}(p_{i-1})$ and the subchains $D_i(1,p_i)$ towered. It can be shown that N has no endpoints. However, $U = \bigcap_{i=1}^{\infty} D_i(1,p_i)^*$ is a proper subcontinuum of X which is evidently and end subcontinuum.

We note that U is in fact the Knaster U-continuum, and the composant C_p of the point $p = \bigcap_{i=1}^{\infty} D_i(p_i)$ consists of U and a ray dense in N joined at their common endpoint p. If we form the quotient space N/U in S^2/U by shrinking U to a point by a monotone map π, where π is a limit of homeomorphisms, as in Section 5.5.3.3, then N/U is also homeomorphic to the Knaster U-continuum with $\pi(U) = u = \pi(p)$ as its only endpoint. In fact, with the embedding given for N in Figure 5.10, N/U in S^2/U is the standard embedding for the U-continuum. In virtue of this fact, N/U has only one prime end E whose impression is all of N/U. Prime end E is defined by an endcut to u. Hence N/U has no s-d-c. We will use N as a model for producing an embedding without a s-d-c of any indecomposable chainable continuum with an end subcontinuum.

In Figure 5.10 the hatched line represents a ray D dense in the end subcontinuum U of N. When U is shrunk to a point, D becomes an endcut to $\pi(U)=u$. That is, a prime end of kind (4b) is carried to a prime end of kind (2b) by the induced map $\bar{\pi}$ on the prime end structures.

5.5.1.3 Example. In Mayer [1980b] and Chapter 2 we gave an example of an indecomposable chainable continuum N', and defining sequence $\{D_i\}$ for N', so that no tower of initial subchains exists (relative to sequence $\{D_i\}$.) Figures 5.11 (a) and (b) illustrate the inverse-limit-with-embedding diagram for N' and shows N' as the intersection of a defining sequence $\{D_i\}$ of convex open disk chain covers. That no other defining sequence for N' has a tower of initial subchains follows from the fact that each composant C of N' is a line composant dense on both ends. That is, C is one-to-one continuous image of the reals such that if p is a point in C, then in the coordinatization of C by the reals, both $(-\infty,p]$ and $[p,\infty)$ are dense in N'. Hence no subcontinuum of N' is an end subcontinuum.

In the embedding pictured in Figures 5.11 (a) and (b), N' has a s-d-c, indicated by the hatched line. We observe that N' can be embedded without a s-d-c. The embedding defined by Figure 5.11(c) has no s-d-c. The hatched line, an endcut to $p = \bigcap_{i=1}^{\infty} D_i(p_i)$, corresponds to the only prime end whose impression is all of N'. Clearly this prime end is of kind (2b). If N' could be used as a model for embedding an indecomposable chainable continuum with no end subcontinua so that there is no s-d-c in the emnedding, then we would have extended the main theorem of this section to all chainable continua. Extending Theorem 5.5.2 would seem to depend upon finding a suitable proper subcontinuum K in chainable indecomposable continuum X so that X could be embedded in S^2 with K the set of principal points of the only prime end whose impression is all of X.

In [1980b] Brechner and Mayer introduced the term principal to describe those embeddings of a nonseparating continuum in S^2 with a s-d-c. A continuum all of whose embeddings into S^2 are principal is termed a principal continuum. Thus we may express the main theorem as:

5.5.2 Theorem

No chainable continuum with at least one end subcontinuum is principal.

5.5.2.1 Remark. The proof of Theorem 5.5.2 combines two techniques: the method of proof of Theorem 1.2 in Mayer, [1980b] (Theorem 2.1.2 in Chapter 2) and the shrinking of a proper subcontinuum K to a point so as to produce in S^2/K an embedding of the quotient continuum X/K with a prime end structure related to that of the original embedding of X in S^2, as detailed in Section 5.2. We will embed X in S^2 in such a way that in the induced embedding of X/K in S^2/K there will be no s-d-c. Because the induced map of prime end structures is one-to-one on prime ends of kind (3b), it will follow that X is embedded in S^2 without a s-d-c.

5.5.3 Proof of Theorem 5.5.2.

Let X be an indecomposable chainable continuum with nondegenerate end subcontinuum K. The proof of Theorem 5.5.2 divides into several lemmas paralleling the proof of

Theorem 2.1.2 in Chapter 2. We state these with little or no proof, as the proofs closely coincide with those in Theorem 2.1.2.

5.5.3.1. Lemma. There exists a defining sequence $\{C_i\}$ of (abstract) chain covers of X satisfying

(1) C_i is taut (that is, the closures of any pair of adjacent links are disjoint).

(2) It takes at least nine links of C_i to span a link of C_{i-1}.

(3) There is a q_i, $1<q_i<h_i$, such that:

(a) $K= \bigcap_{i=1}^{\infty} C_i(1,q_i)^*$,

(b) $C_i(1,q_i)$ covers K minimally.

(4) C_i is at least two-to-one in C_{i-1}; that is, there exist $1 < r_i < s_i < t_i \leq n_i$ such that

(a) $C_i(r_i) \cup C_i(t_i) \subset C_{i-1}(1)$,

(b) $C_i(s_i) \subset C_{i-1}(n_{i-1})$, and

(c) r_i, s_i, t_i are chosen minimally with respect to (a) and (b).

(5) Given i, exactly one of the following conditions obtains for that i:

(a) $1 < q_i \leq r_i < s_i$, or

(b) $1 < r_i < q_i < s_i$.

Proof. Conditions (1) and (2) hold by compactness. Condition (3) holds because K is an end subcontinuum. Condition (4) follows from the indecomposability of X via Lemma 2.6 of Mayer [1980b] (Lemma 2.26 of Chapter 2). Condition (5) follows from the linear order of C_i and the fact that K is proper. For if $q_i \geq s_i$ for infinitely many i, then as $C_i(1,s_i)$ meets every link of C_{i-1}, K would not be proper. We may assume $q_i < s_i$ for all $i \geq 1$). QED

5.5.3.2 The embedding of X in S^2. How C_i might sit in C_{i-1}, the latter straightened out, is shown in Figure 5.12(a), on the assumption condition 5(a) is met. Note that r_i, s_i, and t_i have been chosen minimally.

The sequence $\{C_i\}$ defines X abstractly. However, X can be embedded in S^2 by specifying a defining sequence $\{D_i\}$ of chains of convex open disks such that D_i follows in D_{i-1} the pattern (see Chapter 2 or Mayer [1980b], following Bing [1948]) that C_i follows in C_{i-1}. This is a consequence of a theorem of Bing [1948]. We first think of the D_i chains semi-abstractly, as disks with as yet no embedding in S^2. We then specify how these disks are to be located in S^2 within D_{i-1}. In what follows we assume that condition 5(a) is met by C_i, and hence by D_i. The remaining case is similar.

Suppose D_{i-1} has been realized in S^2 as a chain of convex open disks. Let N_i be an auxiliary sequence of convex open disk chains such that N_i refines D_{i-1} on the model of Figure 5.10(a) with $N_i(1) \cup N_i(p_i) \subset D_{i-1}(q_{i-1})$. We arrange N_i to contain D_i in such a way as to preserve the pattern that D_i follows in D_{i-1}. The three straight-through subchains $N_i(1,j_i)$, $N_i(j_i,p_i,k_i)$, and $N_i(k_i,m_i)$ comprising N_i are illustrated by the dashed lines in Figure 5.12(b).

We embed subchains of D_i as follows:

(1) $D_i(1,r_i)$ ascending in $N_i(1,j_i)$.

(2) $D_i(r_i,s_i)$ descending in $N_i(j_i,k_i)$.

(3) $D_i(s_i,t_i,n_i)$ descending in $N_i(k_i,m_i)$.

Figure 5.12(b) illustrates the resulting embedding of D_i in D_{i-1}. Note that with respect to D_{i-1}, $D_i(1,r_i)$ is descending, but below $D_i(r_i s_i)$; $D_i(r_i,s_i)$ is descending; $D_i(s_i,t_i,n_i)$ is ascending, but below $D_i(1,s_i)$.

We claim that in the embedding of X into S^2 constructed above, S^2-X has no s-d-c. We prove this via Theorem 5.2.5.3 by showing that $S^2/K-X/K$ has no s-d-c, in the embedding of X/K into $S^2/K = S^2$ induced by a monotone map $\pi: S^2 \twoheadrightarrow S^2/K$ that shrinks K to a point in S^2.

5.5.3.3 Construction of the monotone map π. We induce π as the limit of a composition of homeomorphisms of S^2 onto itself, where each homeomorphism successively shrinks K to a smaller set. If $h_1, h_2, \ldots, h_i$ are homeomorphisms such that the composition $h_i \circ h_{i-1} \circ \cdots \circ h_2 \circ h_1$ is defined, then we denote the composition by h_1^i.

We define each $h_i: S^2 \twoheadrightarrow S^2$ as follows: Let $D_0'(1)$ be a convex open disk in S^2 containing D_1^* and let h_i^0 = identity. For each $i \geq 1$, we require that h_i be an orientation preserving homeomorphism satisfying

(i) $h_i(h_1^{i-1}(D_i(1,q_i)^*))$ is a convex open disk $D_i'(1)$ of diameter no more than the mesh of D_i.

(ii) h_i is the identity on the complement of $D_{i-1}'(1)$.

(iii) Each link of chain $h_1^i(D_i(q_i+1,n_i))$ is convex, and of diameter no more than the mesh of D_i.

Note that $D'_i(1)$ meets only link $h^i_1(D_i(q_i+1))$ of chain $h^i_1(D_i(1,q_i+1))$, and $D'_i(1)$ $D'_{i-1}(1)$.

Let D'_i denote the chain of convex open disks whose links are $D'_i(1)$ followed by $h^i_1(D_i(q_i+1,n_i))$, appropriately renumbered. Let $\pi = \lim h^i_1$. Then $\pi: S^2 \twoheadrightarrow S^2/K$ is a monotone map such that the only nondegenerate point inverse is K. We also observe that π is a homeomorphisms on S^2-K; $\pi(X) = X/K = \bigcap_{i=1}^{\infty} D'^{*}_i$; $\pi(K) = k = \bigcap_{i=1}^{\infty} D'_i(1)$; $\{D'_i\}$ is a defining sequence for X/K of chains of convex open disks in $S^2 \cong S^2/K$. It is possible that $\{D'_i\}$ is not minimal at $D'_i(1)$.

Let $\phi : S^2-X \twoheadrightarrow \text{Ext } B$ and $\psi : S^2/K-X/K \twoheadrightarrow \text{Ext } B$ be C-maps, with $\overline{\pi}$ the induced monotone map on the prime end structures via Theorem 5.1.1. The remainder of the proof of Theorem 5.5.2 follows from the lemmas below.

5.5.3.4 Lemma. Point k is an accessible endpoint of X/K; thus there is a prime end E' of $S^2/K-X/K$ corresponding to an endcut to k.

Proof. Given the embedding of Section 5.5.3.3, we can construct a simple canal D converging to a subcontinuum of K as follows: at each stage let D go from the boundary of $D_{i-1}(1)$ between links of N_i and through links of $D_{i-1}(1,q_{i-1})$ to the boundary of $D_i(1)$. One stage of this construction is illustrated in Figure 5.12(b) by the hatched line. Depending upon where in $D_{i-1}(1,q_{i-1})$ link $D_i(1)$ sits, D will converge to K or a proper subcontinuum (possibly degenerate) of K, with transverse crosscuts to X at each point of D.

When K is shrunk to a point, D becomes an endcut $\pi(D)$ to k in $S^2/K-X/K$. Since $k = \bigcap_{i=1}^{\infty} D_i'(1)$, k is an endpoint of X/K. QED

We note that $\{D_i'\}$ has the properties of the defining sequence produced in the proof of Theorem 1.2 in Mayer [1980b] (Theorem 2.1.2 in Chapter 2). That is, the embedding of X/K represented by $\{D_i'\}$ is modelled on the standard embedding of the Knaster U-continuum U in which the composant C_p of endpoint p of U is the only accessible composant. In this embedding of U, there is only one prime end whose impression is all of U, and it is determined by an endcut to p, so is of kind (2b). Continuum X/K shares these properties with U as a result of the embedding we have defined for X, and the action of map π on the defining sequence for X. The following lemmas are sufficiently like those in the proof of Theorem 2.1.2 that we omit the proofs.

5.5.3.5 Lemma. All accessible points of X/K are in the composant C_k of K.

5.5.3.6 Lemma. If F' is a prime end of $S^2/K-X/K$ distinct from E', then I(F') is a proper subcontinuum of X/K contained in C_k.

5.5.3.7 Lemma. Though I(E') = X/K. no prime end of $S^2/K-X/K$ corresponds to a s-d-c.

By Theorem 5.2.5.3, $\overline{\pi}$ is a one-to-one correspondence on prime ends of kind (3b). Hence S^2-X contains no s-d-c. Thus X is nonprincipally embedded in S^2. This concludes the proof of Theorem 5.5.2.

5.5.4 Theorem

If X is an indecomposable chainable continuum with end subcontinuum K, then X has an embedding into S^2 with K containing the set of principal points of the only prime end whose impression is all of X.

Proof. We adopt the embedding and notation of Section 5.5.3. Note that there is a prime end E of S^2-X defined by a simple canal D dense in a subcontinuum of K, as shown in the proof of Lemma 5.5.3.4. Since $\pi(D)$ is an endcut to $\pi(K)=k$ in X/K, and any endcut to k defines the only prime end E' of $S^2/K-X/K$ whose impression is all of X/K, we consequently have $I(E)=X$ and $P(E)\subset K$. So we only need show that for any prime end F of S^2-X such that $I(F)=X$, we have $E=F$.

Let p be any accessible point of X, and let $R\subset S^2-X$ be an endcut to p. Then $\pi(R)$ is an endcut to $\pi(p)$ in $S^2/K-X/K$, so $\pi(p)$ is accessible. By Lemma 5.5.3.5, all the accessible points of X/K lie in composant C_k of k. Hence for the composant C of X containing K, $\pi(C)=C_k$ and p lies in C. Thus X has exactly one accessible composant.

By Illiadis's Theorem (Theorem 3.5 of Brechner [1978]) the prime end structure of S^2-X, namely Bd B under C-map ϕ, consists of a single point f, corresponding to a prime end F of S^2-X such that $I(F)=X$, and an interval (Bd B) - {f}, with the property that each point g in (Bd B)-{f} corresponds to a prime end G such that $I(G)\subset C$. Hence I(G) is properly contained in X.

Since I(E)=X and point f in Bd B is unique in corresponding to a prime end whose impression is all of X, prime ends E and F are equivalent. QED

5.6 Inaccessibility

In conversation, Sam Young asked the following question: Given an indecomposable chainable continuum X and a point p in X, is there an embedding of X into S^2 in which p is accessible? If there is not such an embedding, then point p is inaccessible in every embedding of X into S^2. We have not been able to answer this question in general, but the answer is affirmative in case X is a Knaster continuum. We also show that these is an affirmative answer to the complementary question: Given an indecomposable chainable continuum X and point p in X, is there an embedding of X into S^2 in which p is inaccessible? This cannot be extended to tree-like, or even indecomposable tree-like, continua as we show by examples.

We raise several questions related to Young's concerning the accessibility of subcontinua and composants of indecomposable continua under (re-)embeddings into S^2. These have some bearing on the question as to whether every chainable continuum can be nonprincipally embedded in S^2.

5.6.1 Definition

Let X be a continuum in S^2. A point p in X is essentially accessible iff there is a (re-) embedding of X into S^2 in which p is an accessible point (with respect to some complementary domain of S^2-X.) A composant C of X is essentially accessible iff C contains an essentially accessible point. A continuum K properly contained in X is *essentially accessible* iff there is an embedding of X into S^2 in which K is an accessible continuum (with respect to some complementary domain.) Otherwise p, k, or C is *absolutely inaccessible*.

The concept of essential accessibility parallels, and is suggested by, a concept defined by Brechner in [1978] : a homeomorphism $h:X \twoheadrightarrow X$, where X is a continuum in S^2, is called *essentially extendable to S^2* iff there is an embedding $e:X \to S^2$ such that $ehe^{-1}:e(X) \twoheadrightarrow e(X)$ is extendable to a homeomorphism of S^2 onto itself.

5.6.2 *Theorem*

Let X be a chainable continuum. If point p in X is an endpoint, then p is essentially accessible.

Proof. Since p is an endpoint, a defining sequence $\{C_i\}$ for X exists with p always in a first link. We can require that the defining sequence $\{C_i\}$ be such that each C_i is taut and that it take nine links of C_i to span one link of C_{i-1}. We then specify a defining sequence $\{D_i\}$ of convex open disk chains such that D_i follows in D_{i-1} the pattern C_i follows in C_{i-1}. Embed $\{D_i\}$ in S^2 so that D_i is descending in D_{i-1}. Then an endcut R to $p=\bigcap_{i=1}^{\infty} D_i(1)$ can be constructed. QED

5.6.3 *Examples*

We present below several examples of essential accessibility and absolute inaccessibility.

5.6.3.1 *Example*. Since the pseudo arc is homogeneous, every point of the pseudo arc, and consequently every composant, is essentially accessible. Given any subcontinuum K of the pseudo arc, we can embed the pseudo arc so that some point p in K is accessible, hence K is an accessible continuum.

5.6.3.2 Example. It is not the case that every point of a tree-like plane continuum is essentially accessible, nor even that an endpoint of a tree-like plane continuum is essentially accessible. Let X be the continuum in S^2 formed by a simple triod T and a ray R converging to T as illustrated in Figure 5.13(a). It can be shown that X has a unique embedding into S^2. Each endpoint of T and the endpoint of R are endpoints of X, but clearly every point of T is inaccessible. Since the embedding is unique, every point of T is absolutely inaccessible.

5.6.3.3 Example. There is a nonchainable tree-like continuum in S^2 containing both absolutely accessible and absolutely inaccessible points. Let Y be the continuum formed by a simple triod T and a ray R converging to T as shown in Figure 5.13(b). Let A, B, and C denote the arms of T with junction point p, as illustrated in Figure 5.13(b). Unlike continuum X of Example 5.6.3.2, Y does not have a unique embedding. Figure 5.13(c) shows an embedding of Y which is not equivalent to that of Figure 5.13(b). By a method of proof similar to that used in Chapter 2 and Mayer [1980b] (to show that the sin 1/x continuum has uncountably many inequivalent embeddings into S^2), it can be shown that Y has uncountably many inequivalent embeddings into S^2; we do this by varying the number and pattern of bends that move away from the ends of arms A and C. Each of these embeddings has the same prime end structure, and indeed, the same set of accessible points: $A \cup C \cup R$.

Though Y does not have a unique embedding, all embeddings of Y into S^2 have precisely the same of accessible points. To see that this is the case, divide the ray R into loops L_1, L_2, L_3, L_4, L_5, ... such that L_1 goes around arm B with endpoints near p, L_2 parallels arm C with endpoints near p, and intersects L_1 in one endpoint, L_3 goes around arm B with endpoints (one shared with L_2) near p, L_4 parallels arm A with endpoints near p (one shared with L_3), etc. Then for odd i, $L_i \to B$; for $i=2 \pmod 4$, $L_i \to C$; for $i=0 \pmod 4$, $L_i \to A$.

Since R is linearly ordered, in any reembedding the convergences noted above must be preserved in their proper order after some index i_0. So we may assume that under any reembedding, L_1 is near arm B, L_2 is near arm C, etc.

If in some reembedding of Y, $A \cup C$ contains an inaccessible point q, then every arc to q in the complement of T must intersect R infinitely often. Without loss of generality, suppose q is in arm C. Now the L_i's that converge to C must converge "straight." That is, any bends in a loop other than at a loop end can be straightened out by an isotopy of S^2. Otherwise, the convergence would contradict the linear order on R. Also, loop-ends must converge to the end of arm C, and end points of loops, to junction point p.

Thus the only allowable embeddings of a loop L_i that parallels C in Y are either to parallel or to go around C in the reembedding. Suppose L_i goes around C so that one end point of L_i lies on one side of C and the other on the other side. Loop L_{i+1} must parallel or go around arm B, but given the location of the shared endpoint, this is impossible: L_{i+1} would either first have to go around A, or else backtrack parallel to C. Either alternative violates the fact that loops that converge to C are followed in the linear order by loops that converge solely to B.

Hence in every embedding of Y into S^2, the set of accessible points is $A \cup C \cup R$. Consequently, the set of inaccessible points is $B-\{p\}$. Thus Y contains both absolutely accessible and absolutely inaccessible points.

5.6.3.4 Example. There is a nonchainable tree-like indecomposable continuum in S^2 containing both absolutely accessible and absolutely inaccessible points. Consider again continuum T of Figure 5.4. Arc $[p,s,v]$ of T is accessible at every point in the given embedding, and ray $[s,\infty)$ is inaccessible at every point except s. It can be shown that every embedding of T into S^2 has these two properties. The proof has $[p,s]$ and $[s,v]$ playing the roles of arms A and C in Example 5.6.3.3, and an arbitrary point t in (s,∞) playing the role of arm B. Note that ray $[s,\infty)$ is linearly ordered, and though the convergences are more complicated, we can find a set of loops converging alternately to $[p,s]$ and $[s,v]$ with points between them converging to t. This gives us the requisite characteristics to show that we cannot move loops of $[s,\infty)$ so as to "block" access to arc $[p,s,v]$ or open access to t.

5.6.4 Knaster Continua and Essential Accessibility

We show in this section that Knaster continua have the property that every point is essentially accessible. It follows from Theorem 5.6.5 below, that every point of a Knaster continuum is also essentially inaccessible. Hence there are indecomposable continua in which every point is both essentially accessible and essentially inaccessible.

5.6.4.1 Definition. A Knaster continuum is the intersection of a defining sequence $\{C_i\}$ of chain covers such that each C_i is an n_i-to-one straight through chain in C_{i-1}. The Knaster U-continuum is defined by the intersection of two-to-one straight through chains. If infinitely many n_i's are even, we get a Knaster U-type continuum which has exactly one endpoint. Otherwise, we get a Knaster S-type continuum, which has exactly two endpoints. See Rogers [1970] and Watkins [1980] for further discussions of Knaster contniua.

5.6.4.2. Theorem. Let X be a Knaster continuum. Then every point of X is essentially accessible.

Proof. Let $\{C_i\}$ be a defining sequence of chains for the Knaster continuum X, where each C_i is m_i-to-one in C_{i-1}. By choosing a subsequence, if necessary, we may assume that $m_i \geq 3$, for all i. Suppose p is in X. We may assume that $p=\bigcap_{i=1}^{\infty} C_i(p_i)$, and that p is not in any other link of C_i. We may also assume that C_i is taut and that it takes nine links of C_i to span one link of C_{i-1}. Let $C_i(a_i,p_i,b_i)$ denote the straight-through subchain that contains link $C_i(p_i)$. There are m_i-1 remaining straight-through subchains.

Let $\{D_i\}$ be a defining sequence of chains of convex open disks such that D_i follows in D_{i-1} the pattern C_i follows in C_{i-1}. We will show how to embed X in S^2 as the intersection of the D_i's in such a way that p is accessible.

Embed D_1 as a straight chain in S^2 from left to right. Let A_1 be an arc in $S^2-D_1^*$ with one endpoint a_1 in Bd $D_1(p_1)$, and let A_1 approach D_1 from the top. Embed D_2 in D_1 as follows: embed $D_2(a_2,p_2,b_2)$ on top in D_1. Let $D_2(1,a_2)$ be ascending and below $D_2(a_2,b_2)$; let $D_2(b_2,n_2)$ be descending and below $D_2(1,b_2)$. (We have described Figure 5.14 ((1)+(a) or (2) + (a)) in the case where $m_1=3$.) Let A_2 be an arc in $D_1(p_1)$ from q_1 to a point q_2 in Bd $D_2(p_2)$ meeting no other links of D_1 or D_2. We can so construct A_2 because $D_2(p_2)$ is on top in $D_1(p_1)$ and A_1 meets $D_1(p_1)$ on top.

Suppose D_{i-1} has been embedded in S^2. We may assume D_{i-1} has been straightened out by an orientation preserving homeomorphism of S^2 so that $D_{i-1}(1)$ is on the left. Consider the following conditions on D_i:

(1) $D_i(a_i)\subset D_{i-1}(1)$ and $D_i(b_i)\subset D_{i-1}(n_{i-1})$; that is, $D_i(a_i,b_i)$ is in forward order in D_{i-1}.

(2) $D_i(a_i)\subset D_{i-1}(n_{i-1})$ and $D_i(b_i)\subset D_{i-1}(1)$; that is, $D_i(a_i,b_i)$ is in reverse order in D_{i-1}.

Consider the following conditions on A_{i-1}:

(a) A_{i-1} meets $D_{i-1}(p_i)$ on top.

(b) A_{i-1} meets $D_{i-1}(p_i)$ on bottom.

For a given i, exactly one of the combinations of conditions (1)+(a), (1)+(b), (2)+(a), or (2)+(b) is met. Whether (a) or (b) is met will determine how we embed D_i in D_{i-1}. Then whether (1) or (2) is met will determine whether (a) or (b) is met by A_i. Figure 5.14 shows how D_i is embedded in D_{i-1}, and how A_i meets $D_i(p_i)$ in the case where $m_i=3$ and $D_i(p_i)$ sits in the middle subchain. The general case is handled as follows:

If A_{i-1} meets (a), embed $D_i(a_ip_i,b_i)$ in D_{i-1} on top. Embed $D_i(1,a_i)$ ascending and below $D_i(a_i,b_i)$; embed $D_i(b_i,n_i)$ descending and below $D_i(1,b_i)$. Construct A_i in $D_{i-1}(p_{i-1})$ to q_i in Bd $D_i(p_i)$ extending from endpoint q_{i-1} of A_{i-1}. If D_i meets condition (1), then A_i meets $D_i(p_i)$ on top (because $D_i(a_i,b_i)$ is in forward order in D_{i-1}); hence, A_i meets condition (a). Alternately, if D_i meets condition (2), then A_i meets condition (b). In either case we are ready for the next stage of the embedding.

Alternately, if A_{i-1} meets condition (b), embed $D_i(a_i,p_i,b_i)$ in D_{i-1} on bottom. Embed $D_i(1,a_i)$ descending and above $D_i(a_i,b_i)$; embed $D_i(b_i,n_i)$ ascending and above $D_i(1,b_i)$. Construct A_i in $D_{i-1}(P_{i-1})$ to q_i in Bd $D_i(p_i)$ extending from endpoint q_{i-1} of A_{i-1}. If D_i meets condition (1), then A_i meets $D_i(p_i)$ on bottom, and so satisfies condition (b). Alternately, if D_i meets condition (2), then A_i meets condition (a). In either case we are ready for the next stage of the embedding.

We follow the above procedure inductively, producing an embedding of X into S^2. Since Int $A_i \subset D_{i-1}(p_{i-1})$, $A=\bigcup_{i=1}^{\infty} A_i$ is an endcut to p in X. Hence in the embedding we have constructed, p is an accessible point. QED

5.6.4.3 Remark. It can be shown that in the embedding of X produced in Theorem 5.6.4.2 every point of the composant C_p of point p is accessible, as is every proper subcontinuum of X contained in C_p. Since every point is essentially accessible, every composant and every proper subcontinuum of a Knaster continuum is essentially accessible also.

5.6.5 Theorem

Let X be indecomposable chainable continuum and p a point in X. Then there is an embedding of X into S^2 in which p is an inaccessible point.

Proof. Let $\{C_i\}$ be a defining sequence of chains for X. We may assume:

(i) C_i is taut, and it takes nine links of C_i to span one link of C_{i-1}.

(ii) C_i is at least three-to-one in C_{i-1}; that is there are $1<r_i<s_i<n_i$ such that each subchain $C_i(1,r_i)$, $C_i(r_i,s_i)$, and $C_i(s_i,n_i)$ runs at least from one end of C_{i-1} to the other.

(iii) $p = \bigcap_{i=1}^{\infty} C_i(p_i)$ with p in no other link of C_i.

(iv) Exactly one of the following conditions obtains:

(1) $1 \leq p_i < r_i$

(2) $r_i \leq p_i < s_i$

(3) $s_i < p_i \leq n_i$

Let D_i be the convex open disk chain sequence chosen as usual. We will embed the D_i chains in S^2 so that p is inaccessible in the resulting embedding of X.

Consider the following conditions on that subchain among $D_i(1,r_i)$, $D_i(r_i,s_i)$, and $D_i(s_i,n_i)$ which contains link $D_i(p_i)$:

(a) The subchain runs through D_{i-1} in forward order.

(b) The subchain runs through D_{i-1} in reverse order.

(For the distinction between forward and reverse order see conditions (1) and (2) in the proof of Theorem 5.6.4.) Depending upon which combination of conditions (1), (2), or (3), and (a) or (b) is satisfied by D_i, we embed D_i in D_{i-1} in one of the six pattern indicated in Figure 5.15. Of course, the three chosen subchains of D_i may not be straight-through as indicated in the diagrams. In that case, regard the diagram as representing a consolidation of the actual subchains, with the actual subchains embedded as follows:

(1) $D_i(1,s_i)$ descending, and $D_i(s_i,n_i)$ ascending and above $D_i(1,s_i)$. (See Figure 5.15 (1) + (a) and (1) + (b).)

(2) D_i descending. (See Figure 5.15 (1) + (a) and (2) + (b).)

(3) $D_i(1,s_i)$ descending, and $D_i(s_i,n_i)$ ascending and between $D_i(1,r_i)$ and $D_i(r_i,s_i)$. (See Figure 5.15 (3) + (a), (3) + (b).)

We claim that in the resulting embedding of X in S^2, no endcut to p exists in S^2-X. Suppose that A is an endcut to p. We may assume that A meets only link $D_i(p_i)$, for all i. Suppose that A meets $D_1(p_1)$. Consulting the diagrams in Figure 5.15, we see that it is impossible for A to meet $D_2(p_2)$ without meeting other links of D_1, unless $D_1(p_1)$ happens to be an endlink. In case $D_1(p_1)$ is an endlink, consulting diagrams (1) and (3) of Figure 5.15, we see that $D_2(p_2)$ cannot be an endlink and still have A meet no other links of D_1. Since $D_2(p_2)$ is not an endlink, A cannot meet $D_3(p_3)$ without meeting links of D_2 other than $D_2(p_2)$. Hence p is inaccessible in the given embedding. QED

5.6.6 Concluding Remarks and Questions

Theorem 5.6.5 shows that every point of an indecomposable chainable continuum is inaccessible in some embedding. In parallel with the definition of essentially accessible, we can say that a point p of a plane continuum X is essentially inaccessible if there is some embedding of X into S^2 in whiich p is an inaccessible point. If there is no such embedding, then p is accessible in every embedding, and so absolutely accessible. In Example 5.6.3.3, point s of continuum T is absolutely accessible. Every boundary point of a locally connected plane continuum is absolutely accessible (with respect to some complementary domain.)

It would be interesting to know the answers to the following questions:

5.6.6.1 Question. (Sam Young) Is every point of a chainable continuum essentially accessible?

5.6.6.2 Question. Is every proper subcontinuum (every composant) of a chainable continuum essentially accessible?

The answer to 5.6.6.1 could be negative, while the answer to 5.6.6.2 is affirmative, since every point accessible in some embedding implies that every proper subcontinuum and every composant is accessible in some embedding. The implication does not necessarily go the other way. An affirmative answer to either 5.6.6.1 or 5.6.6.2 would likely lead to the extension of Theorem 5.5.2 to all chainable continua.

5.6.6.3 Question. Is there a tree-like nonchainable indecomposable continuum in S^2 in which every point is essentially inaccessible?

5.6.6.4 Question. Is there a tree-like nonchainable indecomposable continuum in S^2 with an absolutely inaccessible composant?

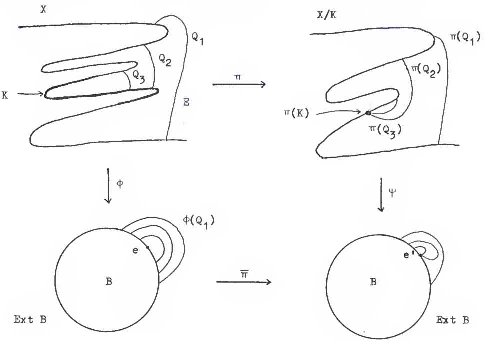
We conjecture that the endpoint composants in the X-odic continuum of Mayer [1980a] and Chapter 3 is such a continuum. The X-odic continuum is nonchainable and atriodic. There are probably triodic examples. If all embeddings of continuum T of Example 5.6.3.4 have the same set of accessible points, then all but two composants of T are absolutely inaccessible. The accessible composants of T are the composant of arc [p,s,v] and the outer limiting composant of T (not shown in Figure 4).

5.6.6.5 Question. Is there an indecomposable plane continuum which contains an absolutely inaccessible point, but every composant of which is essentially accessible?

5.6.6.6 Question. If X is a principal continuum, does X have to contain an absolutely inaccessible point? (composant?)

5.6.6.7 Question. Suppose X is an indecomposable continuum in S^2. Is it the case that X contains an absolutely inaccessible composant iff X contains an absolutely accessible composant?

5.6.6.8 Remark. As the final form of this dissertation was being prepared, E. Duda showed us a simple example which answers Question 5.6.6.1 in the negative for decomposable chainable containua. We can modify the example so as to show the answer is also negative in the case where the chainable containuum is in decomposable. Let C be the middle third cantor set. Let I = [o,1]. The decomposable example is C x I with pairs of endpoints X I of internals deleted at odd stages in the construction of C identified at 1 in I and pairs of endpoints x I of intervals deleted at even stages identified at 0 in I. For the indecomposable example, we shrink the above set x to a point p in E^2. Then we construct a Knaster U-continuum U with p as its endpoint. Reversing the shrinking produces the indecomposable example. The point $(c,\frac{1}{2})$, where c is not an endpoint of a deleted interval, is inaccessible in every embedding.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Q_2
Q_3
K
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$\pi(K)$
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ϕ
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Ext B
$\overline{\pi}$
e'
B
Ext B

Figure 5.1

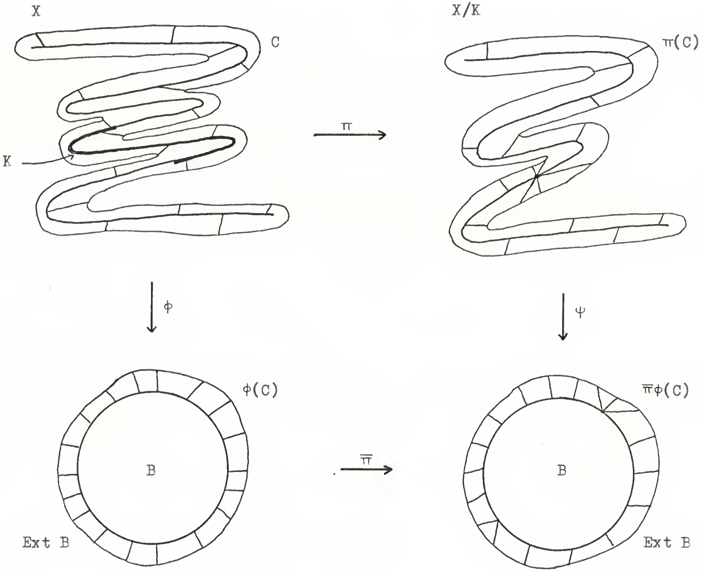


Figure 5.2

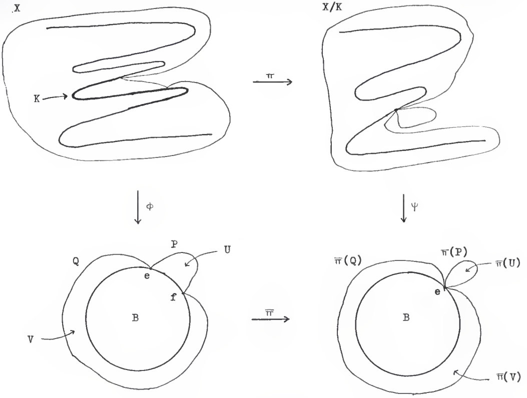
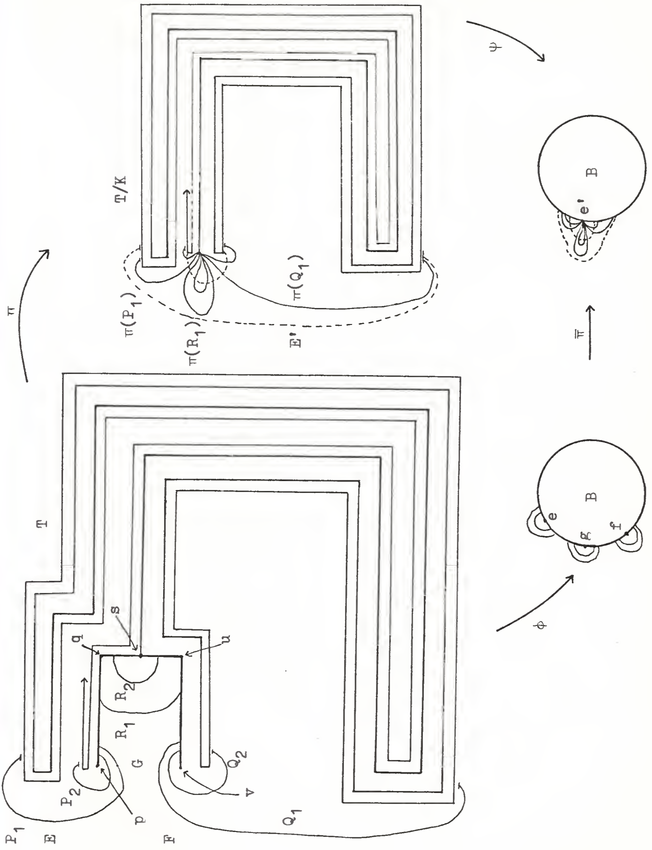


Figure 5.3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Q_2
Q_1
F
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E'
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B
e
g
f

Figure 5.4

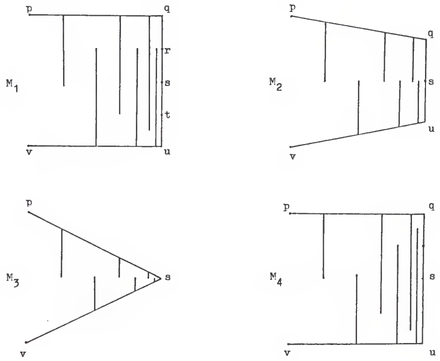


Figure 5.5

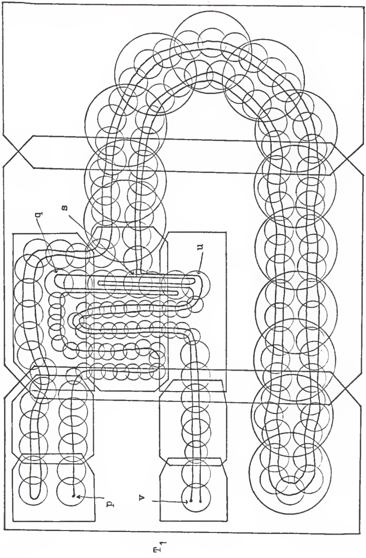
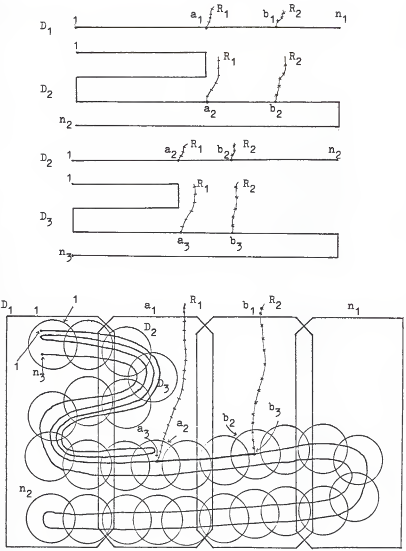


Figure 5.6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R_2
n_1
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R_1
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R_2
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R_1
b_1
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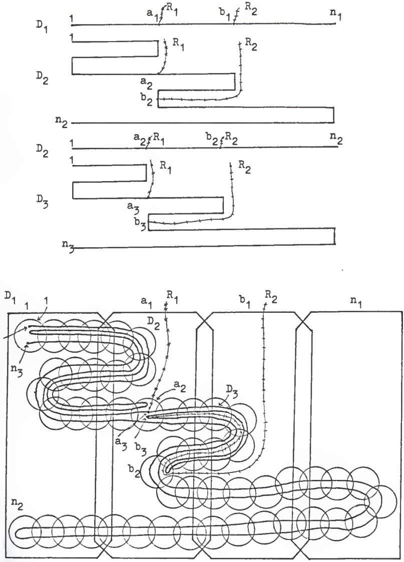
Figure 5.7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b_2
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R_1
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b_3
n_3
D_1
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a_1
R_1
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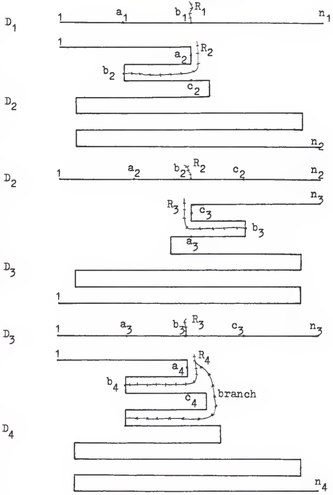
b_2
n_2

Figure 5.8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b_2
R_2
c_2
n_2
n_3
R_3
c_3
b_3
a_3
D_3
1
D_3
1
a_3
b_3
R_3
c_3
n_3
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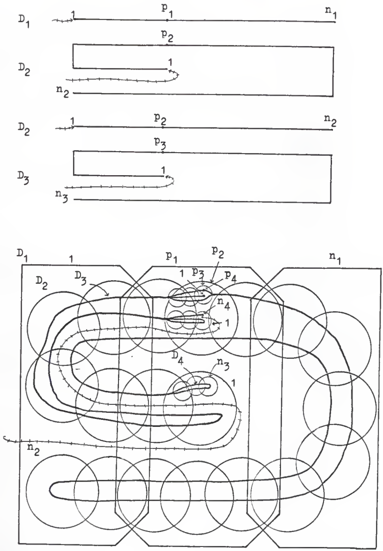
R_4
a_4
b_4
branch
c_4
D_4
n_4

Figure 5.9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n1
p2
D2
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n2
D2
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p2
n2
p3
D3
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n3
D1
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p1
p2
n1
D2
D3
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p3
p4
n4
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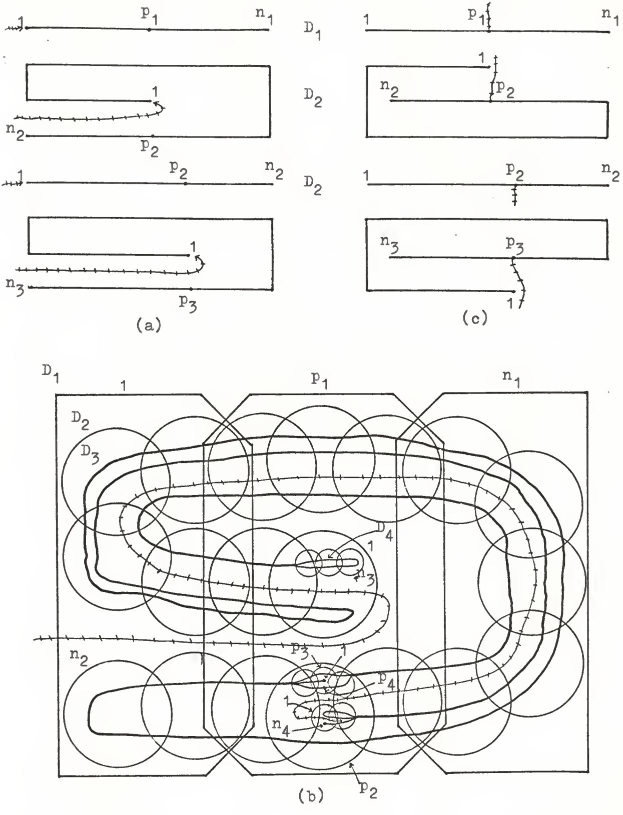
D4
n3
1
n2

Figure 5.10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p1
n1
D1
1
n2
p2
D2
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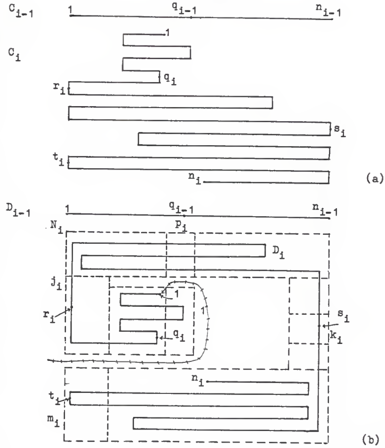
p2
n2
D2
1
n3
p3
(a)
1
p1
n1
1
n2
p2
1
p2
n2
n3
p3
1
(c)
D1
1
p1
n1
D2
D3
D4
1
n3
n2
p3
1
p4
1
n4
p2
(b)

Figure 5.11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q_{i-1}
n_{i-1}
C_i
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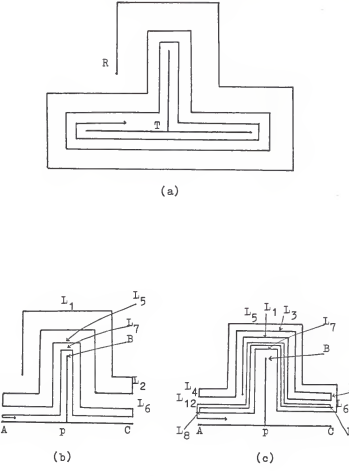
q_i
r_i
s_i
t_i
n_i
(a)
D_{i-1}
1
q_{i-1}
n_{i-1}
N_i
p_i
D_i
j_i
1
1'
r_i
s_i
q_i
k_i
n_i
t_i
m_i
(b)

Figure 5.12



(a)

(b)

(c)

Figure 5.13

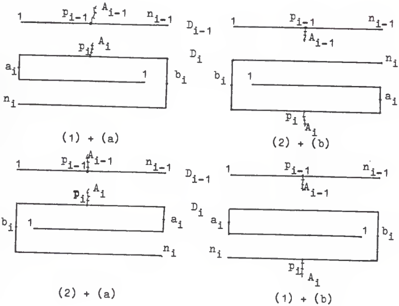
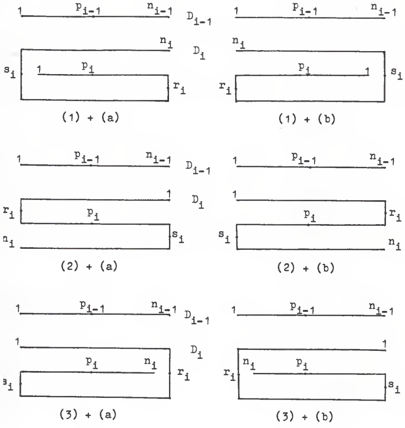


Figure 5.14



(1) + (a)   (1) + (b)

(2) + (a)   (2) + (b)

(3) + (a)   (3) + (b)

Figure 5.15

CHAPTER 6
PRINCIPAL EMBEDDINGS OF PLANE CONTINUA AND EXTENDABLE HOMEOMORPHISMS

6.1 Introduction

In Chapters 2 and 3 we gave examples of principal embeddings of chainable continua, and nonchainable atriodic continua in S^2. In this chapter we give examples of principal embeddings of triodic nonseparating plane continua. Several of the examples would appear to be principal continua, though we have no proof of that conjecture. Because of the symmetry with which the examples are constructed, they admit periodic homeomorphisms. The periodic homeomorphisms can be extended to S^2, and as might be expected, interchange the channels in S^2 corresponding to the simple dense canals. However, these triodic examples do have the fixed point property, as for each $\varepsilon>0$ there is an ε-retraction of the continuum to an n-od. We devote Section 6.2 to the presentation of these examples.

In Section 6.3, we show that there are chainable continua, principally embedded into S^2, which admit periodic homeomorphisms. These homeomorphisms are also extendable to S^2, and interchange the channels corresponding to the s-d-c's. The chainable examples arise from a modification of the construction in Lewis [1981].

6.2 Principal Embeddings of Triodic Continua.

A triodic nonseparating continuum in S^2 may be tree-like, that is, without interior, or may have a nonempty interior. We confine ourselves to the tree-like case.

6.2.1 Definition

Let p be a point in an indecomposable continuum X. We say that p is a point of generalized order n ($1<n \leq \omega_0$ or $n=c$) iff in every sufficiently small neighborhood U of p, p is a cut point of order n in the component of p in U.

6.2.2 Example

For each n, $3 \leq n <\infty$, there is a tree-like nonchainable indecomposable continuum M_n in S^2, embedded with exactly n s-d-c's. Furthermore, M_n admits a period n homeomorphism which is extendable to S^2. Figures 6.1 and 6.2 illustrate the inverse-limit-with-embedding diagram for M_3 and M_4, respectively. The same diagram is used for each stage of the construction. It can be seen that M_n is the intersection of a defining sequence $\{T_i\}$ of n-od covers whose links are convex open disks. Each T_i is the union of n chains $\{C_i^j\}_{j=1}^n$ united at their common first link, and consisting of the same number of links (n_i) in each, following symmetric patterns in the containing cover T_{i-1}. That is, C_i^1 refines each of C_{i-1}^1, C_{i-1}^2, ..., C_{i-1}^n two-to-one, in that order. In C_{i-1}^1 it is innermost, in C_{i-1}^2 it is next innermost, and so on, until in C_{i-1}^n it is outermost, with its last link in the junction link of T_{i-1}. The remaining C_i^i's are placed symmetrically.

Note that M_n has a single point p of generalized order n which is the intersection of the junction links of the T_i's. In the given embedding, point p and its composant are inaccessible. The composant C_p of p is the union of n rays, each dense in M_n, united at their common endpoint p.

In Figure 6.1, the hatched line represents the first state in the construction of a s-d-c D in M_3. Ray D enters the junction link of T_1 near $C_2^3(n_2)$, between the C_1^3 and C_1^2 "arms" of T_1, proceeds through links of C_1^2 and C_1^1, between links of T_2, and enters the junction link of T_2 between the C_2^3 and C_2^1 arms of T_2. Since D meets every link of two arms, and since each arm meets every link of the preceeding stage, D is dense in M_3. The other two s-d-c's are similarly constructed, starting near $C_2^2(n_2)$ and $C_2^1(n_2)$ in the junction link of T_1.

Since the construction of M_n is symmetric, there is a period n homeomorphism of M_n induced by rotating the arms at each stage cyclicly. It is clear from the construction that this homeomorphism is extendable to a homeomorphism of S^2, indeed to a period n homeomorphism of S^2.

It is perhaps worth noting that though M_n has a point of generalized order n, it is the only "unusual" point of M_n, for M_n has neither endpoints, nor any other point of generalized order different from two. With the exception of point p, every point of M_n is an interior point of an arc.

We can write C_p as the union of an increasing tower $\{K_i\}$ of simple n-ods. If $C_i^k(q_i)$ is the first link of C_i^k in $C_{i-1}^k(n_{i-1})$, then K_i is the component containing p of

$C_p \cap (\bigcup_{k=1}^{n} C_i^k(1,q_i)^*)$. Note that K_i consists of n arms, one going down each arm of the cover T_{i-1}. Hence there is a retraction r_i of M_n to K_i that moves no point more than the mesh of T_{i-1}. So for any $\varepsilon > 0$, there is an i' such that $r_{i'}$ retracts M_n to $K_{i'}$, moving no point more than . From this it follows that M_n has the fixed point property.

6.2.3 Example

We can modify the construction in Example 6.2.1 to produce a continuum M_c embedded in S^2 with $c=2^{\omega_0}$ s-d-c's such that M_c admits a homeomorphism of period 2^n, for every n, each extendable to S^2. Continuum M_c has the further properties that it contains a single point p of generalized order c which is the common endpoint of uncountably many rays, each dense in M_c, whose union is the composant C_p of p. Both p and C_p are inaccessible in the given embedding, and every other point of M_c is an interior point of an arc.

Figure 6.3 shows the first three stages in the construction of M_c as the intersection of a defining sequence $\{T_i\}$ of 2^{i-1}-od tree covers. At each stage, we double the number of chains united at a common junction link in cover T_i. Of each pair of chains (C_i^1,C_i^2), (C_i^3,C_i^4), ..., $(C_i^{2^i-1}, C_i^{2^i})$ we require that both elements of the pair follow the same pattern, while successive pairs follow a pattern symmetric to the others, each pair refining all but one arm of T_{i-1}. As the variously marked lines in Figure 6.3 indicate, we can construct an infinite binary tree dense in M_c in such a way that each branch constitutes a s-d-c.

That M_c admits a period 2^n homeomorphism, extendable to S^2, is clear from the symmetry of the construction.

6.2.4 Example

We can modify the construction in Example 6.2.2 to produce a continuum N_n, $3 \leq n < \infty$, so that N_n is embedded in S^2 with exactly n s-d-c's, admitting an extendable period n homeomorphism, but with a central point of countably infinite generalized order, and countably infinitely many endpoints. Continuum N_n is the intersection of a defining sequence $\{T_i\}$ of ni-od tree covers. Cover T_i consists of $n(i-1)$ chains which go straight out each of the $n(i-1)$ arms of T_{i-1}, and n chains which refine every arm of T_{i-1} two-to-one, united at a common junction link. The "long" chains of T_i wrap counterclockwise around the "short" chains of T_i.

Figure 6.4 illustrates the construction of N_3 for the first few stages. The hatched line represents one of the three s-d-c's in S^2-N_3. Point p, the intersection of the tower of junction links, is of countably infinite generalized order. Note that the composant C_p of p consists of three rays, each dense in N_3, and countably infinitely many arcs (of increasing length) whose union is dense in N_3, united at common endpoint p. Since the canal parallels a long arm at each stage, and each long arm meets every link of the preceeding stage, each of the three simple canals is dense in N_3, and so is a s-d-c.

6.2.5 Example

In Example 6.2.4, the long arms of each cover T_i refine every arm of T_{i-1} two-to-one. We can construct a continuum X_n, $1 \leq n < \infty$, embedded in S^2 with exactly n s-d-c's, but in which no arm of T_i refines every arm of T_{i-1}. In refining arms of T_{i-1}, the long arms of T_i wrap alternately clockwise and counterclockwise, in contrast to Example 6.2.4. In fact, an arm of T_i refines at most 1/n arms of T_{i-1}. As is the case with Example 6.2.4, X_n admits an extendable period n homeomorphism, has a central point of countably infinite generalized order, and has countably infinitely many endpoints.

Continuum X_n is the intersection of a defining sequence $\{T_i\}$ of ni-od covers. Figure 6.5 illustrates our construction for X_3. Of the 3i chains which comprise T_i, 3i-3 of them go straight out the 3(i-1) arms of T_{i-1}. The remaining three chains each refine two-to-one one third of the arms of T_{i-1}. Thus each cover T_i consists of mostly "short" chains and three "long" chains. Cover T_i refines T_{i-1} in such a way that the long chains of T_i each refine a long chain of T_{i-1} first, and then refine one third of the remaining (short) chains of T_{i-1}, by going around the short chains of T_i. The order in which the long chains of T_i refine the arms of T_{i-1} alternates between clockwise (cw) and counterclockwise (ccw.)

Figure 6.6 illustrates X_3 as the intersection of defining sequence of 3i-od covers by convex open disks in S^2. The hatched line in both Figures 6.5 and 6.6 represents one of the three s-d-c's in S^2-X_3. Note that canal D always parallels one of the long chains of T_i as the canal goes through links of T_{i-1}. Using this fact, we can show that D is dense in X_3.

First we specify the defining sequence for X_3 more precisely. Cover T_1 is the union of three chains of convex open disks denoted C_1^1, C_1^2, C_1^3. The three chains are united at a common junction link $T_1(1)$, which is the first link of each chain. We place T_1 in S^2 with $T_1(1)$ at the origin and the three chains (arms of T_1) radiating outwards, evenly spaced, C_1^1 downward, and in ccw order about $T_1(1)$.

To avoid writing "C" with superscripts and subscripts repeatedly, we will denote chain C_j^k of cover T_j by k:j henceforth.

Cover T_2, refining T_1, consists of six chains, 1:2, 2:2, 3:2, 4:2, 5:2, and 6:2, united at junction link $T_2(1)$ in $T_1(1)$ at the origin. Chains 1:2, 2:2, and 3:2 go straight out the arms of T_1 with the same first index, and their second links meet $T_2(1)$ in the same ccw order about the origin that the arms of T_1 meets $T_1(1)$. The second links of the long chains 4:2, 5:2, and 6:2 meet $T_2(1)$ as follows: 4:2 between 1:2 and 2:2, 5:2 between 2:2 and 3:2, and 6:2 between 3:2 and 1:2. Refer to the "clock" of Figure 6.7, and note that the chains are represented only by their first index. Observe that 4:2, 5:2, and 6:2 emanate from $T_2(1)$ ccw of, but adjacent to, chains 1:2, 2:2, and 3:2, respectively. For k=2,1,0, chain 6-k:2 refines arm 3-k:1 by going straight out, around the last link of 3-k:2, and straight back to the junction link. Note that 6-k:2 refines one third of the arms of T_1 by wrapping cw.

Suppose that T_2 has been straightened out by an orientation preserving homeomorphism of S^2 that leaves the origin fixed so that the arms of T_2 radiate outward, evenly spaced, and with 1:2 downward. Cover T_3 consists of nine chains 1:3 through 9:3 united at junction link $T_3(1)$ placed at the origin. The first six chains go straight out the arms of T_2 with the same first index, and so are in the same ccw order about the origin as the six chains of T_2. As the clock in Figure 6.7 indicates, the long chains 7:3, 8:3, and 9:3 are placed so that their second links meet the junction link cw of, but adjacent to the second links of 4:3, 5:3, and 6:3, respectively. For k=2,1,0, chain 9-k:3 refines one third of the arms of T_2 by wrapping ccw: straight out 6-k:2 and back, then straight out 3-k:2 and back, terminating in the junction link near the second link of 9-(k-1):3 (addition in k is done mod 3.) Note that 9-k:3 first refines a long chain of T_2, then one third of the short chains, ending with a next-to-longest, in ccw order about the origin.

In general, suppose cover T_{i-1} has been realized in S^2, as a 3(i-1)-od cover of convex open disks. Suppose that T_{i-1} has been straightened out by an orientation preserving homeomorphism of S^2 that leaves the origin fixed, so that the arms of T_{i-1} radiate outward, evenly spaced, and with chain 1:i-1 downward. Junction link $T_i(1)$ is placed at the origin. We place the first 3(i-1) arms of T_i straight out the arms of T_{i-1} with the same first index, and with their second links meeting the junction link in the same ccw order about the origin as the arms of T_{i-1}.

For k=2,1,0, the long chain 3i-k:i meets the junction link adjacent to 3(i-1)-k:i, for even i, ccw of it, and for odd i, cw of it. Thus in Figure 6.7, 10:4, 11:4, and 12:4 are ccw of 7:4, 8:4, and 9:4, respectively, while 13:5, 14:5, and 15:5 are cw of 10:5, 11:5, and 12:5, respectively. Note that long chain 3i-k:i always meets the junction link between 3(i-1)-k:i (which goes straight out a long chain of T_{i-1}) and 3(i-2)-k:i (which goes straight out a long chain of T_{i-2}.)

Long chain 3i-k:i refines one third of the arms of T_{i-1} two-to-one by going around the short chains of T_i. First 3i-k:i refines 3(i-1)-k:i, then one third of the shorter chains. For even i, 3i-k:i wraps cw about the origin in refining T_{i-1}. For odd i, 3i-k:i wraps ccw. Figure 6.8 illustrates the "i-clock" for odd i. The arms of T_{i-1} that 3i-k:i refines are those in ccw order between 3(i-1)-k:i-1 and 3(i-2)-(k-1):i-1, inclusive (k+1 for even i, k-1 for odd i, mod 3).

We claim that long chain 3i-k:i meets every link of cover T_{i-3}. Figure 6.9 diagrams how the chains of T_i refine those of T_{i-1} for the first few stages. The order from left to right corresponds to ccw. Regard Figure 6.9 as wrapped around a drum; we have repeated the first few left hand columns on the right for convenience. By tracing the lines showing the refinement relations, our claim can be seen to be true for the first few stages.

For the general case, we will show that long chain 3i-1:i meets every link of T_{i-3} for odd $i \geq 5$. The remaining cases follow by symmetry. Observe that on the i-clock of Figure 6.8, _all_ chains of T_{i-3} are represented, partitioned into three equal sets by the numbers 3i-2, 3i-1, and 3i. Of course, the first indices of chains of T_{i-3} only run up to 3(i-3), but the chains of T_{i-2} and T_{i-1} with the same first indices go straight out them.

Consider the chains of T_{i-1}, i-3 in number, that lie strictly between 3(i-1)-1:i-1 and 3(i-2):i-1 in ccw order. Each of these chains goes straight out an arm of T_{i-2} which itslef goes straight out an arm of T_{i-3}. The collection of arms of T_{i-3} thus singled out is one of the three sets into which we have partitioned the arms of T_{i-3}. Since i is odd, 3i-1:i refines T_{i-1} ccw, so meets every link of the chains of T_{i-1} between 3(i-1)-1:i-1 and 3(i-2):i-1 inclusive, and so certainly meets those strictly between. Hence 3i-1:i meets all links of one third of the chains of T_{i-3}, the _second_ set of our partition.

Now 3(i-1)-1:i-1 is a long chain, and 3i-1:i meets every link of it. However, i-1 is even, so 3(i-1)-1:i-1 refines cw, from 3(i-2)-1:i-2 to 3(i-3)-2:i-2 inclusive. These chains of T_{i-2} go straight out the chains of T_{i-3} in the first set of our partition. Thereby 3i-1:i meets every link of the _first_ set of the partition of the arms of T_{i-3}. Finally, chain 3i-1:i meets every link of 3(i-2):i-1. The latter goes straight out 3(i-2):i-2, which is a long chain of T_{i-2}. Since i-2 is again odd, 3(i-2):i-2 refines

T_{i-3} ccw. This is the same direction on the i-clock as 3i-1:i refines, but starting where the latter leaves off. Since 3(k-2):i-2 meets every link of the arms of T_{i-3} in the third set of our partition, 3i-1:i meets them as well. Thus we have shown 3i-1:i meets every link of T_{i-3}.

It follows that a canal which always parallels a long chain at each stage is dense in X_3. Since the endlinks of long chains lie in the junction link, but alternate cw and ccw, we can construct a s-d-c D by paralleling the long chain at each stage in reverse order.

One can generalize the above construction to produce continuum X_n for each $n \geq 1$. If $\{T_i\}$ is a defining sequence for X_n, then each long chain of T_i will meet every link of T_{i-n}. We embed X_n in S^2 in a fashion similar to that for X_3, so that S^2-X_n has exactly n s-d-c's, and there is an extendable period n homeomorphism of X_n.

6.2.6 Remark

Continua M_n, M_c, X_n, and N_n all have the fixed point property, since for any $\varepsilon>0$, we can find a tree, an m-od for some m, in the composant C_p of central point p, to which the continuum retracts, without moving any point more than ε.

6.3 Extendable Homeomorphisms of Principally Embedded Chainable Continua

In [1981] W. Lewis constructs, for each n, $2 \leq n < \infty$, a chainable continuum Y_n admitting an extendable period n homeomorphism. What is surprising and insightful about Lewis's construction is that while Y_n is constructed as the intersection of n-od tree-covers, the tree covers admit consolidations to chains with the diameter of the links remaining small.

Continuum Y_n is embedded in S^2 without a s-d-c. However, it requires only a slight modification of Lewis's construction to produce a chainable continuum Y'_n embedded in S^2 with exactly n s-d-c's, and admitting an extendable period n homeomorphism.

Figure 6.10 shows the modification required to be made to Figure 1 of Lewis [1981]. Essentially we have lengthened each branch of the 3-od cover of Y_3 at each stage, so that the final link of each arm lies in the junction link of the preceding stage, rather than in the final link of the corresponding arm. Continuum Y'_3 is the intersection of a defining sequence $\{T_i\}$ of 3-od covers, in each of which we carry out the above modification to Lewis's construction. The consolidation of T_2 to a chain is indicated in Figure 6.10 by the lighter markings joining branches of the tree cover. The line marked with circles is one of the three s-d-c's in $S^2-Y'_3$.

As is the case with Lewis's original constructions, each cover can be made crooked in the preceding one, so that the intersection is chainable, but is now an embedding of the pseudo arc.

Since the constructions are symmetric in S^2, Y'_n admits an extendable homeomorphism of period n. On S^2 the extension interchanges the channels in $S^2-Y'_n$ down which the s-d-c's go. By making the covers crooked, we show that the pseudo arc admits extendable periodic homeomorphisms interchanging the channels of s-d-c's.

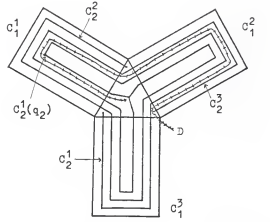


Figure 6.1

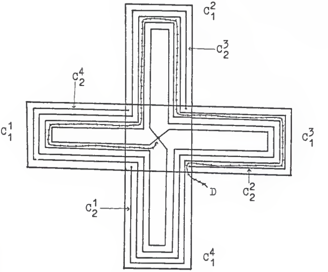
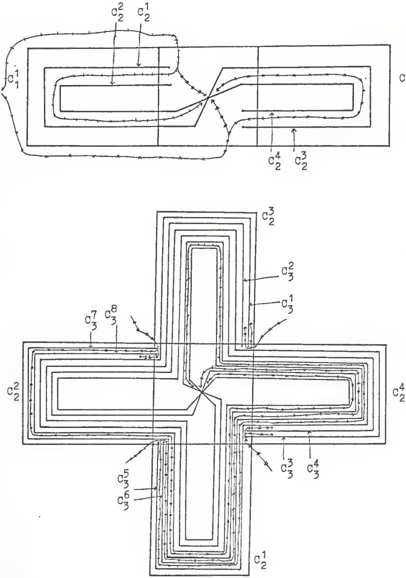


Figure 6.2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c_2^4
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c_3^7
c_3^8
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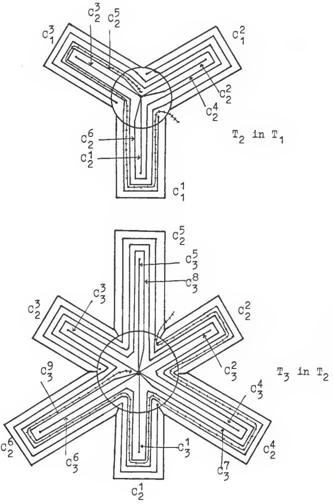
c_2^4
c_3^5
c_3^6
c_3^3
c_3^4
c_2^1

Figure 6.3


c_1^3
c_2^3
c_2^5
c_1^2
c_2^2
c_2^4
c_2^6
c_2^1
T_2 in T_1
c_1^1
c_2^5
c_3^5
c_3^8
c_3^3
c_2^3
c_2^2
c_3^9
c_3^2
T_3 in T_2
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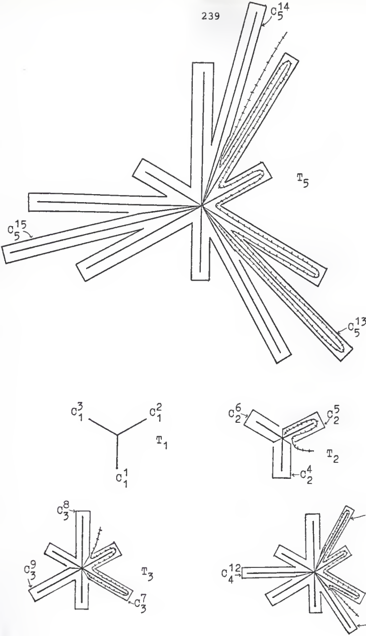
c_2^6
c_3^6
c_3^1
c_3^7
c_2^4
c_2^1

Figure 6.4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T_5
C_5^{15}
C_5^{13}
C_1^3
C_1^2
T_1
C_1^1
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C_2^5
T_2
C_2^4
C_3^8
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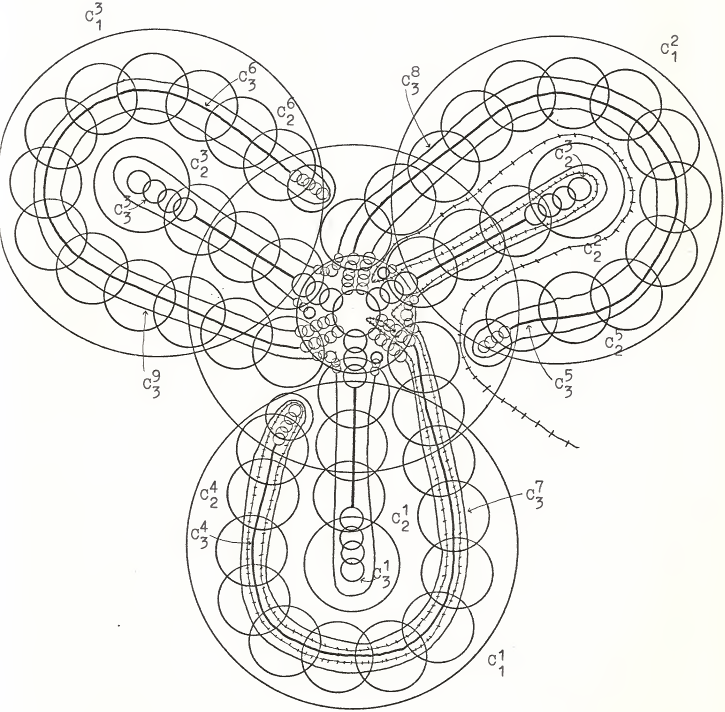
T_3
C_3^7
C_4^{11}
C_4^{12}
T_4
C_4^{10}

Figure 6.5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C_2^5
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C_1^1

Figure 6.6

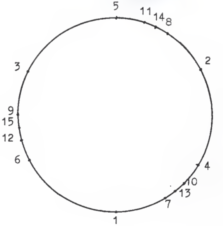


Figure 6.7

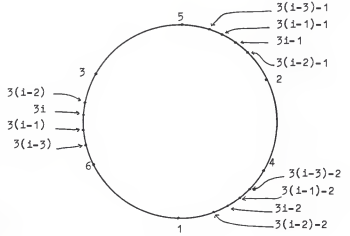
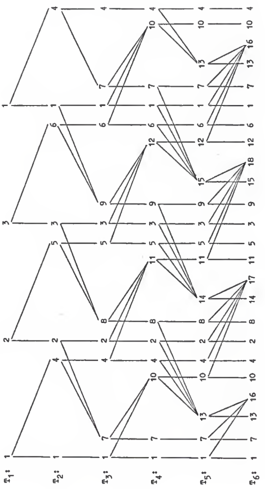


Figure 6.8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T_2: 1 4 2 5 3 6 1 4
T_3: 1 7 4 2 8 5 3 9 6 1 7 4
T_4: 1 7 10 4 2 8 11 5 3 9 12 6 1 7 10 4
T_5: 1 7 13 10 4 2 8 14 11 5 3 9 15 12 6 1 7 13 10 4
T_6: 1 7 13 16 10 4 2 8 14 17 11 5 3 9 15 18 12 6 1 7 13 16 10 4

Figure 6.9

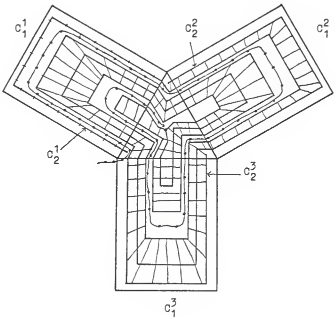

Figure 6.10

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Beverly L. Brechner

Beverly L. Brechner,
Chairman, Associate
Professor of
Mathematics

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

J. Jay Zeman
Professor of Philosophy

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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